MATHEMATICAL FOUNDATIONS FOR PARTICIPATION ONTOLOGIES

CARMEN CHUI AND MICHAEL GRÜNINGER

Abstract. The notion of participation as a relation between objects, activities, and time has been axiomatized in various ontologies. In this paper, we focus on three of these ontologies – PSL-Core, Gangemi’s axioms, and DOLCE. We provide a verification of these participation ontologies by introducing ontologies for new classes of mathematical structures known as incidence bundles and incidence foliations. The new mathematical ontologies serve as reusable ontology design patterns for participation, and also are the basis for mappings between the different participation ontologies. Finally, we illustrate the concept of ontology transfer through the use of these ontology design patterns.

Contents

1. Existing Participation Ontologies 2
2. Basic Approach to Participation 3
2.1. Motivating Scenario I 3
2.2. Incidence Structures 4
2.3. Bundles 5
2.4. Root Theory \textit{T}_{in\_bundle} 8
2.5. \textit{T}_{partial\_bundle} 11
2.6. \textit{T}_{present\_bundle} 12
2.7. \textit{T}_{wp1\_bundle} 14
2.8. \textit{T}_{nip\_bundle} 16
2.9. \textit{T}_{ip\_bundle} 19
2.10. \textit{T}_{pp\_bundle} 21
2.11. \textit{T}_{plane\_flag\_bundle} 23
2.12. \textit{T}_{line\_flag\_bundle} 24
2.13. \textit{T}_{flag\_bundle} 26
2.14. Summary 27
3. Incidence Bundles and Participation Ontologies 27
3.1. Verification of \textit{T}_{psl\_participation} 27
3.2. Verification of \textit{T}_{gangemi} 30
3.3. Beyond Incidence Bundles 31
4. Incidence Foliations 32
4.1. Motivating Scenario II 32
4.2. Introducing Foliations 32
4.3. Root Theory \textit{T}_{in\_foliation\_root} 32
4.4. \textit{T}_{downward\_in\_foliation} 34
4.5. \textit{T}_{upward\_in\_foliation} 36

Date: March 19, 2014.
1. Existing Participation Ontologies

In this work, we examine the notion of participation in three ontologies: the Process Specification Language (PSL), the Descriptive Ontology for Linguistic and Cognitive Engineering (DOLCE), and the set of OWL axioms provided by Aldo Gangemi for participation.

The Process Specification Language (PSL) is an ontology designed to facilitate the correct and complete exchange of process information among manufacturing systems. These applications include scheduling, process modelling, process planning, production planning, simulation, and project management. The PSL ontology is organized into PSL-Core and a set of partially ordered extensions; the core ontology consists of four disjoint classes: activities can have zero or more occurrences, activity occurrences begin and end at time points, time points constitute a linear ordered set with end points at infinity, and objects are elements that are not activities, occurrences, or time points. In PSL, the ternary relation, participates_in(x, o, t), is used to specify that an object x participates in an activity occurrence o at a time point t. In other words, an object can participate in an activity occurrence only at those time points at which both the object exists and the associated activity is occurring.

As the first module of the WonderWeb library of foundational ontologies, DOLCE aims at capturing the categories which underlie natural language and human common sense. DOLCE is based on the distinction between enduring and perduring entities, which are also referred to as continuants and occurrents, where the fundamental difference between the two is related to their behaviour in time. Endurants are wholly present at any time: they are observed and perceived as a complete concept, regardless of a given snapshot of time. Perdurants, on the other hand, extend in time by accumulating different temporal parts, so they are only partially present at any given point in time. In DOLCE, endurants are involved in an occurrence, so the notion of participation is not considered parthood. Rather, participation is time-indexed in order to account for the varieties of participation in time, such as temporary participation and constant participation.

References
In the OWL axiomatization provided by Aldo Gangemi, the notion of participation is represented as a simple binary relation between objects and events. The participation relation does not include any temporal indexing.

How are these three ontologies related? Do they capture the intuitions of participation? Do the actual models of these ontologies correspond to their intended models? Is there a common ontology design pattern that can be used to tie them together? If so, what is this pattern and how can ontology designers apply it in practice? In the subsequent sections, we show how these various notions of participation are related together and present a different approach to applying ontology patterns in a mathematical setting.

2. Basic Approach to Participation

For the sake of simplicity, we provide a motivating scenario that illustrates the common sense notion of participation. Depending on the context, not all objects involved in a given process exist and/or participate in certain parts of the process. In this section, we show how the axiomatization of participation in PSL-Core and Gangemi’s design pattern can be verified using a new class of mathematical structures known as incidence bundles.

2.1. Motivating Scenario I. Before we get into detail about the notion of basic participation, we provide an everyday scenario that outlines how objects, activity occurrences, and time points play a role in participation. Take, for example, the process of mixing ingredients to bake cookies: we have the dry ingredients (flour, baking soda, sugar), wet ingredients (eggs), and the process of blending these ingredients together. Two blending processes are involved: mixing the dry ingredients together, and then mixing the dry with wet ingredients.

In Figure 1, we depict the relationships among the activity occurrences, objects, and time points in the baking example. There are two activity occurrences, $o_1$ and $o_2$, which represent the mixing of the dry ingredient (flour), and the mixing of the dry and wet ingredients (flour and eggs), respectively. The objects (ingredients) involved in these tasks are $x_1$ (flour) and $x_2$ (eggs). The time points associated with each ingredient indicate their existence in time: $x_1$ exists at $t_1$ and $t_2$, and $x_2$ exists at $t_2$ and $t_3$. In the figure, bold lines denote incidence and dotted lines denote participation: for example, the flour ingredient ($x_1$) exists at time $t_1$, the activity of dry ingredient mixing ($o_1$) occurs at time $t_1$, and the flour participates in the mixing activity at this time point.

From Figure 1 we can extract several conditions that we need to capture formally:

1. There are three disjoint sets of elements;
2. Pairs of elements are related by an presence relation;
3. Triples of elements are related by participation, which vary across time points.

In the remainder of this section, we will explore the mathematical structures which formalize these conditions.
objects/lines

activity occurrence/planes

time/points

Figure 1. Sample figure of incidence bundles found in a baking task. $o_1$ and $o_2$ represent the occurrences of baking tasks, $x_1$ and $x_2$ represent the objects (ingredients) involved in the baking tasks, and $t_1$, $t_2$, and $t_3$ represent the time points associated with the baking tasks. Bold lines denote incidence and dotted lines denote participation.

2.2. Incidence Structures. To formalize the first two conditions, we begin with the notion of incidence structures [1], first used to generalize intuitions from geometry.

**Definition 1.** A k-partite incidence structure is a tuple $I = (\Omega_1, \ldots, \Omega_k, \text{in})$, where $\Omega_1, \ldots, \Omega_k$ are sets with $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$

$$\text{in} \subseteq \bigcup_{i \neq j} \Omega_i \times \Omega_j$$

Two elements of $I$ that are related by $\text{in}$ are called incident.

The neighbourhood of an element is the set of its incident elements:

$$N(x) = \{y : (x, y) \in \text{in}\}$$

Since we want structures with three disjoint classes of elements, we will be considering tripartite incidence structure[4], we will refer to the elements of the sets as points, lines, and planes. We can consider Figure 1 to be a depiction of a tripartite incidence structure in which planes are interpreted as objects, lines as activity occurrences, and points as time. Two elements are incident if they are joined by a solid line; the interpretation of incidence corresponds to the presence of objects and activity occurrences at the timepoint.

---

4All axioms discussed in this paper can be found in COLORE (Common Logic Ontology Repository). Ontologies in COLORE are organized into hierarchies, which are sets of ontologies with the same signature [5].

The axioms for tripartite incidence structures are in the $\mathbb{H}^{\text{tripartite\_incidence}}$ Hierarchy, which can be found at colore.oor.net/tripartite\_incidence/
2.3. Bundles. For the class of mathematical structures that will formalize the third condition, we need to take a closer look at the intuitions about the ternary relation that are at the heart of the notion of participation. In particular, objects participate in an activity occurrence relative to timepoints. If we interpret objects by planes, activity occurrences by lines, and timepoints by points, then we need a notion of “temporary incidence”, that is, an incidence relation on planes and lines that varies points. In particular, a bipartite incidence structure is specified on the set of planes and lines that are incident with a point. To formalize this property, we will generalize the notion of bundle from the field of algebraic topology [6]:

Definition 2. A bundle is a triple \( \langle X, \pi, Y \rangle \), where \( \pi : X \rightarrow Y \) is a map (called the projection of the bundle) on the structures \( X \) (called the base space) and \( Y \) (called the total space). The set \( \pi^{-1}(y) \) is called the fibre of the bundle over each \( y \in Y \).

With participation, the base space is the tripartite incidence structure, and we want a fibre of the bundle to be a bipartite incidence structure which represents the condition that an object is participating in an activity occurrence at a particular timepoint.

Definition 3. Suppose \( \mathbb{I} = \langle P, L, Q, \text{in} \rangle \in \text{Mod}(T_{\text{nonempty point}}) \).

Suppose for any \( p \in P \),

\[
I_p \subseteq (N(p) \cap L) \times (N(p) \cap Q)
\]

\[
J = \langle P \cup L \cup Q, \bigcup_{p \in P} I_p \rangle
\]

The bundle \( \langle J, \pi, \mathbb{I} \rangle \) is a \( T \)-incidence bundle iff \( \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle \) is definably equivalent to some \( \mathcal{M} \in \text{Mod}(T) \), where \( T_{\text{weak bipartite}} \leq T \).

\( I_p \) is a set of pairs of planes and lines that are all incident with the point \( p \). In a \( T \)-incidence bundle, this set of pairs is definably equivalent to a bipartite incidence structure as axiomatized by the theory \( T \) which is an extension of the root theory \( T_{\text{weak bipartite}} \). The incidence structure in the definition must be a model of \( T_{\text{nonempty point}} \) to guarantee that \( \pi^{-1}(p) \neq \emptyset \).

Theorem 1. Let \( T \) be a theory in \( \mathbb{H}_{\text{bipartite incidence}} \).

For any set of models \( \mathcal{M}_1, \ldots, \mathcal{M}_n \in \text{Mod}(T) \), there exists a unique \( T \)-incidence bundle \( \langle J, \pi, \mathbb{I} \rangle \) such that for each fibre, \( \langle N(p) \cap Q, N(p) \cap L, \pi^{-1}(p) \rangle \) is definably equivalent to some \( \mathcal{M}_i \).

Proof. Suppose \( \mathcal{M}_1, \ldots, \mathcal{M}_n \in \text{Mod}(T) \), where \( T_{\text{weak bipartite}} \leq T \).

Suppose \( \mathcal{M}_i = \langle L_i, Q_i, \text{in}_i \rangle \), for each \( 1 \leq i \leq n \).

Let

\[
L = \bigcup_i L_i
\]

\[
Q = \bigcup_i Q_i
\]

Since the incidence relation in a bipartite incidence structure is symmetric and reflexive, we consider the symmetric reflexive closure of the relation \( I_p \) in the definition of incidence bundle.
Claim 1: \( \mathbb{I} = (P, L, Q, \text{in}) \in M_{\text{nonempty point}}. \)

Proof. It is easy to see that \( P \neq \emptyset \).

The \( \mathcal{M}_i \) are bipartite incidence structures, so \( L_i \cap Q_i = \emptyset \), and hence \( L \cap Q = \emptyset \).

Since the elements in \( P \) are not in the domain of any \( \mathcal{M}_i \), we have \( P \cap L = \emptyset \) and \( P \cap L = \emptyset \).

It is easy to see from the definition that the \( \text{in}^{P_i} \) relation is symmetric and reflexive, that distinct elements of \( P \) are not incident, and that distinct elements of \( L \) are not incident.

Let \( J = (P \cup L \cup Q, \bigcup_{p_i \in P} \text{in}^{P_i}) \)

Let \( \pi : J \to \mathbb{I} \) be a mapping such that \( \pi(\langle x, y \rangle) = p_i \) for each \( \langle x, y \rangle \in \text{in}^{P_i} \).

Claim 2: \( \langle J, \pi, \mathbb{I} \rangle \) is a \( T \)-incidence bundle.

Proof. We have already shown that \( \mathbb{I} \) is a tripartite incidence structure, so that \( \langle J, \pi, \mathbb{I} \rangle \) is a bundle.

Suppose \( \mathcal{M}_i = (L_i, Q_i, \text{in}^{\mathcal{M}_i}) \). Consider the bijection \( \varphi : (N(p_i) \cap L, N(p_i) \cap Q, \pi^{-1}(p_i)) \to (L_i, Q_i, \text{in}^{\mathcal{M}_i}). \)

If we consider the fibres
\[
\pi^{-1}(p_i) = \{\langle x, y \rangle : \langle x, y \rangle \in J\}
\]
\[
= \{\langle x, y \rangle : \langle x, y \rangle \in \text{in}^{P_i}\}
\]
we can see that
\[
\langle x, y \rangle \in \pi^{-1}(p_i) \iff \langle \varphi(x), \varphi(x) \rangle \in \text{in}^{\mathcal{M}_i}
\]
and hence \( (N(p_i) \cap L, N(p_i) \cap Q, \pi^{-1}(p_i)) \) is definably equivalent to \( \mathcal{M}_i \in \text{Mod}(T). \)

Theorem 1 shows how we can construct an incidence bundle by assembling a set of bipartite incidence structures. The next theorem provides an alternative characterization by showing how to construct an incidence bundle from an arbitrary tripartite incidence structure.

Definition 4. Let \( \mathbb{I} = (\Omega_1, \ldots, \Omega_n, \text{in}) \) be an incidence structure. A flag of \( \mathbb{I} \) is a set of elements of \( \Omega_1 \cup \ldots \cup \Omega_n \) that are mutually incident. A flag \( \mathbf{F} \) is maximal if there is no element \( x \in \Omega \setminus \mathbf{F} \) such that \( \mathbf{F} \cup \{x\} \) is also a flag.

Unfortunately, we cannot guarantee that maximal flags exist in all tripartite incidence structures that we consider. However, we can generalize the notion of flag to the following:
Definition 5. Let \( \mathbb{I} = (P, L, Q, \text{in}) \) be a tripartite incidence structure.
\[ F \text{ is a near-flag complex in } \mathbb{I} \text{ iff } \]
\[ F = \{ (x, y, z) : \langle z, x \rangle, \langle z, y \rangle \in \text{in}, x \in Q, y \in L, z \in P \} \]

We also have the following result which shows how closely related near-flags are to flags within tripartite incidence structures.

Theorem 2. If \( T_{\text{weak\_tripartite}} \leq T \), there exists a surjection \( \varphi : \text{Mod}(T) \rightarrow \text{Mod}(T \cup T_{\text{point\_cover}}) \) such that the near flags in \( \mathcal{M} \) are equivalent to the flags in \( \varphi(\mathcal{M}) \).

Proof. \( \square \)

Near flags within a tripartite incidence structure play a key role because they can be used to construct the incidence bundles:

Theorem 3. Suppose \( \mathbb{I} \in \mathfrak{M}_{\text{weak\_tripartite}} \).
\( \langle F, \pi, \mathbb{I} \rangle \) is a \( T_{\text{weak\_bipartite}} \)-incidence bundle iff \( F \) is a near-flag complex for \( \mathbb{I} \).

Proof. \( \Leftarrow \): Suppose \( \mathbb{I} \in \mathfrak{M}_{\text{weak\_tripartite}} \) and \( F \) is a near-flag complex for \( \mathbb{I} \).
\( (x, y, z) \in F \) iff \( \langle z, x \rangle, \langle z, y \rangle \in \pi^{-1}(z) \)
Thus
\[ x \in N(z) \cap Q \]
\[ y \in N(z) \cap L \]
Let
\[ \pi^{-1}(z) = I_z = \{ (x, y) : (x, y, z) \in F \} \]
Consider the bijection
\[ \varphi : (N(z) \cap L, N(z) \cap Q, I_z) \rightarrow (X, Y, \text{in}^M) \]
such that
\[ (x, y) \in I_p \Leftrightarrow (\varphi(x), \varphi(y)) \in \text{in}^M \]
Since
\[ N(z) \cap L = \emptyset, N(z) \cap Q = \emptyset \]
\[ I_z \subseteq N(z) \cap Q \times N(z) \cap L \]
we can see that \( \mathcal{M} = (X, Y, \text{in}^M) \in \text{Mod}(T_{\text{weak\_bipartite}}) \).
\( \langle N(z) \cap L, N(z) \cap Q, I_z \rangle \) is definably equivalent to \( \mathcal{M} \) and hence \( \langle F, \pi, \mathbb{I} \rangle \) is a \( T_{\text{weak\_bipartite}} \)-incidence bundle.
\( \Rightarrow \): Suppose \( \mathbb{I} \in \mathfrak{M}_{\text{weak\_tripartite}} \) and \( \langle F, \pi, \mathbb{I} \rangle \) is a \( T_{\text{weak\_bipartite}} \)-incidence bundle.
\( (x, y, z) \in F \) iff
\[ (z, x), (z, y) \in \pi^{-1}(z) \]
By the definition of incidence bundle,
\[ x \in N(z) \cap Q \]
\[ y \in N(z) \cap L \]
so that
\[ (z, x), (z, y) \in \text{in} \]
and hence \( F \) is a near-flag complex. \( \square \)
Theorem 3 shows not only that incidence bundles exist, but it also characterizes the set of all possible incidence bundles that can exist for a given tripartite incidence structure.

2.4. **Root Theory** \( T_{in\text{-bundle}} \). Given the notion of incidence bundle, we can now explore the hierarchy of theories which axiomatize different classes of incidence bundles. We begin with the root theory of the hierarchy.

2.4.1. **Introducing** \( M_{in\text{-bundle}} \). Suppose \( \langle P, L, \text{point}, \text{line}, \text{in} \rangle \in M_{weak\text{-bipartite}} \). We will use the following notation:

\[
P^* = \{ p : p \in P, N(p) \cap P \neq \emptyset \}
\]

\[
L^* = \{ l : l \in L, N(l) \cap L \neq \emptyset \}
\]

In other words, \( P^* \) is the set of nonisolated points in a bipartite incidence structure.

**Definition 6.** \( M \in M_{in\text{-bundle}} \) iff \( M = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \) such that

1. \( I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in} \rangle \in M_{weak\text{-tripartite}} \);
2. \( \langle J, \pi, I^* \rangle \) is a \( T_{weak\text{-bipartite}} \)-incidence bundle such that

\[
\langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p)
\]

The \( \text{tin} \) relation corresponds to the fibres of the incidence bundle; in Figure [1] this is denoted by triples of elements that are adjacent by dashed lines:

\[
\{ \langle o_1, x_1, t_1 \rangle, \langle o_2, x_2, t_2 \rangle \}
\]

These sets denote the fibres in the incidence bundle; in particular, we have

\[
(o_1, x_1) \in \pi^{-1}(t_1), (o_2, x_2) \in \pi^{-1}(t_2)
\]

Note that structures in \( M_{in\text{-bundle}} \) may contain isolated points and lines, but these cannot be elements of an incidence bundle in such structures.

2.4.2. **Axiomatization of Incidence Bundles.** The axioms for the theory \( T_{in\text{-bundle}} \) can be seen in Figure [2]. This is the root theory of the \( H_{in\text{-bundle}} \) Hierarchy of incidence bundles (see Figure [3]).

**Proposition 1.** \( T_{in\text{-bundle}} \) is consistent.

**Proof.** A model constructed by Mace4 can be found at colore.oor.net/incidence_bundle/consistency/output/in_bundle.model

2.4.3. **Representation Theorem for Mod(T_{in bundle}).**

**Theorem 4.** \( M \in \text{Mod}(T_{in\text{-bundle}}) \) iff \( M \in M_{in\text{-bundle}} \).

**Proof.** \( \Rightarrow \)

Suppose \( M \in \text{Mod}(T_{in\text{-bundle}}) \).

Since

\[
T_{in\text{-bundle}} \models T_{weak\text{-bipartite}}
\]

condition (1) in the definition of \( M_{in\text{-bundle}} \) is satisfied.

For each point \( p \in P \), let

\[
R_p = \{ \langle x, y \rangle : \langle x, y, p \rangle \in \text{tin} \}
\]
(1) \((\forall x, y, p) \ tin(x, y, p) \supset plane(x) \land line(y) \land point(p)\)

(2) \((\forall x, y, p) \ tin(x, y, p) \supset in(p, x) \land in(p, y)\)

(3) \((\forall x, y) \ in(x, y) \supset in(y, x)\)

(4) \((\forall x) \ (point(x) \lor line(x) \lor plane(x)) \supset in(x, x)\)

(5) \((\forall x) \ point(x) \supset \lnot line(x)\)

(6) \((\forall x, y) \ point(x) \land point(y) \land in(x, y) \supset (x = y)\)

(7) \((\forall x, y) \ line(x) \land line(y) \land in(x, y) \supset (x = y)\)

(8) \((\forall x) \ point(x) \supset \lnot plane(x)\)

(9) \((\forall x) \ plane(x) \supset \lnot line(x)\)

(10) \((\forall x, y) \ plane(x) \land plane(y) \land in(x, y) \supset (x = y)\)

**Figure 2.** \(T_{in\text{-}bundle}\): Axioms of the root theory of the hierarchy \(\mathbb{H}_{incidence\text{-}bundle}\).

Let 
\[
\mathcal{J} = \langle P \cup L \cup Q, \bigcup_{p \in P} R_p \rangle 
\]

If we define a mapping \(\pi : \mathcal{J} \rightarrow \mathcal{I}\) by
\[
\pi(R_p) = p
\]

then \(\langle \mathcal{J}, \pi, \mathcal{I} \rangle\) is a bundle.

**Claim:** \(\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle\) is definably equivalent to any model of \(T_{weak\text{-}bipartite}\).

**Proof.** It is easy to see that 
\[
N(p) \cap L \cap Q = \emptyset
\]

By Axioms 1 and 2
\[
\bigcup_{p \in P} (R_p) \subseteq \bigcup_{p \in P} (N(p) \cap Q \times (N(p) \cap L))
\]
so that

\[ R_p \subseteq (N(p) \cap Q) \times (N(p) \cap L) \]

Thus, condition (2) in the definition of \( H^{in\text{-bundle}} \) is satisfied, and \( \mathcal{M} \in H^{in\text{-bundle}} \).

Suppose \( \mathcal{M} \in H^{in\text{-bundle}} \).

**Axiom [1]**

By condition 2 in Definition [6]

\[ \langle x, y, p \rangle \in tin \iff \langle x, y \rangle \in \pi^{-1}(p) \]

Since \( J \) is a \( T_{weak\text{-bipartite}} \)-incidence structure, by Definition [3] we have

\[ x \in (N(p) \cap Q) \]
\[ y \in (N(p) \cap L) \]

Thus, if \( \langle x, y, p \rangle \in tin \), then

\[ \langle x \rangle \in \text{line} \]
\[ \langle y \rangle \in \text{plane} \]
\[ \langle p \rangle \in \text{point} \]
so that
\[ M \models (\forall x, y, p) \ tin(x, y, p) \supset plane(x) \land line(y) \land point(p) \]

**Axiom 2**

By condition 2 in Definition 6,
\[ \langle x, y, p \rangle \in \tin \iff \langle x, y \rangle \in \pi^{-1}(p) \]

Since \( J \) is a \( T_{\text{weak,bipartite}} \)-incidence structure, by Definition 3, we have
\[ x \in (N(p) \cap Q) \]
\[ y \in (N(p) \cap L) \]

Since
\[ N(p) \cap L = \{ l : \langle p, l \rangle \in \text{in} \} \]
\[ N(p) \cap Q = \{ q : \langle p, q \rangle \in \text{in} \} \]

if \( \langle x, y, p \rangle \in \tin \), then
\[ \langle p, x \rangle, \langle p, y \rangle \in \text{in} \]

\[ M \models (\forall x, y, p) \ tin(x, y, p) \supset \text{in}(p, x) \land \text{in}(p, y) \]

By condition (2) in the definition of \( \mathfrak{M}^{\text{in,bundle}} \) and the Satisfiability Theorem for \( \mathfrak{M}^{\text{weak,tripartite}} \), we have
\[ M \models T_{\text{weak,tripartite}} \]

so that the remaining axioms \( T_{\text{in,bundle}} \) are satisfied. \( \square \)

This result not only shows that incidence bundles exist, but they also characterize structures in \( \text{Mod}(T_{\text{in,bundle}}) \) up to isomorphism – it shows us how to construct all possible models of \( T_{\text{in,bundle}} \) by choosing a set of substructures of a weak tripartite incidence structure. \( T_{\text{in,bundle}} \) is therefore a verified ontology.

**Corollary 1.** \( T_{\text{in,bundle}} \) is a conservative extension of \( T_{\text{weak,tripartite}} \).

**Proof.** \( \square \)

2.5. \( T_{\text{partial,bundle}} \).

2.5.1. Axiomatization of \( T_{\text{partial,bundle}} \).

**Definition 7.**
\[ T_{\text{partial,bundle}} = T_{\text{in,bundle}} \cup T_{\text{partial,tripartite}} \]

**Proposition 2.** \( T_{\text{partial,bundle}} \) is consistent.

**Proof.** A model constructed by Mace4 can be found at [colore.oor.net/incidence_bundle/consistency/output/partial_bundle.model](colore.oor.net/incidence_bundle/consistency/output/partial_bundle.model) \( \square \)
2.5.2. Introducing \(\mathcal{M}_{\text{partial bundle}}\).

**Definition 8.** \(\mathcal{M} \in \mathcal{M}_{\text{partial bundle}}\) iff 
\[ \mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in, tin} \rangle \] such that 
1. \(I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in} \rangle \in \mathcal{M}_{\text{partial tripartite}}\); 
2. \(\langle J, \pi, I^* \rangle\) is a \(T_{\text{weak bipartite}}\)-incidence bundle such that 
\[ \langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p) \]

**Lemma 1.** \(\mathcal{M}_{\text{partial bundle}} \subseteq \mathcal{M}_{\text{in bundle}}\)

**Proof.** Suppose \(\mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in, tin} \rangle \in \mathcal{M}_{\text{partial bundle}}\). Since 
\(\mathcal{M}_{\text{partial tripartite}} \subseteq \mathcal{M}_{\text{weak tripartite}}\) by Theorem 4, 
\(I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in} \rangle \in \mathcal{M}_{\text{weak tripartite}}\). 
By Definition 8, \(\mathcal{M} \in \mathcal{M}_{\text{in bundle}}\). □

**Theorem 5.** Suppose \(\mathcal{I} \in \mathcal{M}_{\text{partial tripartite}}\).
\(\langle F, \pi, \mathcal{I} \rangle\) is a \(T_{\text{weak bipartite}}\)-incidence bundle iff \(F\) is a near-flag complex for \(\mathcal{I}\).

**Proof.** This follows from Theorem 3 and Lemma 1. □

2.5.3. Representation Theorem for \(\text{Mod}(T_{\text{partial bundle}})\).

**Theorem 6.** \(\mathcal{M} \in \text{Mod}(T_{\text{partial bundle}}), \) iff \(\mathcal{M} \in \mathcal{M}_{\text{partial bundle}}\).

**Proof.** \(\Rightarrow\): Suppose \(\mathcal{M} \in \text{Mod}(T_{\text{partial bundle}})\).
Since 
\(T_{\text{partial bundle}} \models T_{\text{partial tripartite}}\)
\(\mathcal{M} \in \mathcal{M}_{\text{partial tripartite}}\) and condition (1) in Definition 8 is satisfied. 
Since 
\(T_{\text{partial bundle}} \models T_{\text{in bundle}}\)
by Theorem 4, \(\mathcal{M} \in \mathcal{M}_{\text{in bundle}}\) and condition (2) in Definition 8 is satisfied. 
Thus \(\mathcal{M} \in \mathcal{M}_{\text{partial bundle}}\).
\(\Leftarrow\): Suppose \(\mathcal{M} \in \mathcal{M}_{\text{partial bundle}}\).
By condition (1) in Definition 8,
\[ \mathcal{M} \models T_{\text{partial tripartite}}\]
By Lemma 1 and Theorem 4
\[ \mathcal{M} \models T_{\text{in bundle}}\]
Combining these together we have 
\[ \mathcal{M} \models T_{\text{partial bundle}} \]
□

**Corollary 2.** \(T_{\text{partial bundle}}\) is a conservative extension of \(T_{\text{partial tripartite}}\).

**Proof.** □

2.6. \(T_{\text{present bundle}}\).
2.6.1. Axiomatization of $T_{\text{present bundle}}$.

Definition 9.

$$T_{\text{present bundle}} = T_{\text{in bundle}} \cup T_{\text{partial flag}}$$

Proposition 3. $T_{\text{present bundle}}$ is consistent.

Proof. A model constructed by Mace4 can be found at colore.or.net/incidence_bundle/consistency/output/present_bundle.model

2.6.2. Introducing $\mathcal{M}_{\text{present bundle}}$.

Definition 10. $\mathcal{M} \in \mathcal{M}_{\text{present bundle}}$ iff

$$\mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle$$

such that

1. $I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in} \rangle \in \mathcal{M}_{\text{partial flag}}$;
2. $\langle J, \pi, I^* \rangle$ is a $T_{\text{weak bipartite}}$-incidence bundle such that

$$\langle q, l, p \rangle \in \text{tin} \iff (q, l) \in \pi^{-1}(p)$$

Lemma 2.

$$\mathcal{M}_{\text{present bundle}} \subseteq \mathcal{M}_{\text{in bundle}}$$

Proof. Suppose $\mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathcal{M}_{\text{present bundle}}$. Since

$$\mathcal{M}_{\text{partial flag}} \subseteq \mathcal{M}_{\text{weak tripartite}}$$

$I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in} \rangle \in \mathcal{M}_{\text{weak tripartite}}$.

By Definition 10, $\mathcal{M} \in \mathcal{M}_{\text{in bundle}}$.

Theorem 7. Suppose $I \in \mathcal{M}_{\text{partial flag}}$.

$$\langle F, \pi, I \rangle$$ is a $T_{\text{weak bipartite}}$-incidence bundle iff $F$ is a near-flag complex for $I$.

Proof. This follows from Theorem 3 and Lemma 2.

2.6.3. Representation Theorem for $\text{Mod}(T_{\text{present bundle}})$.

Theorem 8. $\mathcal{M} \in \text{Mod}(T_{\text{present bundle}})$, iff $\mathcal{M} \in \mathcal{M}_{\text{present bundle}}$.

Proof. $\Rightarrow$)

Suppose $\mathcal{M} \in \text{Mod}(T_{\text{present bundle}})$.

Since

$$T_{\text{partial bundle}} \models T_{\text{partial flag}}$$

$\mathcal{M} \in \mathcal{M}_{\text{partial flag}}$ and condition (1) in Definition 8 is satisfied.

Since

$$T_{\text{present bundle}} \models T_{\text{in bundle}}$$

by Theorem 4, $\mathcal{M} \in \mathcal{M}_{\text{in bundle}}$ and condition (2) in Definition 10 is satisfied.

Thus $\mathcal{M} \in \mathcal{M}_{\text{present bundle}}$.

$\Leftarrow$)

Suppose $\mathcal{M} \in \mathcal{M}_{\text{present bundle}}$.

By condition (1) in Definition 10

$$\mathcal{M} \models T_{\text{partial flag}}$$

By Lemma 2 and Theorem 4

$$\mathcal{M} \models T_{\text{in bundle}}$$
Combining these together we have
\[ M \models T_{\text{present\_bundle}} \]

2.7. \textit{T\textsubscript{wpl\_bundle}.}

2.7.1. Axiomatization of \textit{T\textsubscript{wpl\_bundle}}.

**Definition 11.** \textit{T\textsubscript{wpl\_bundle}} is the set of axioms in \textit{T\textsubscript{present\_bundle}} together with \[(\forall x,p) \text{ plane}(x) \land \text{in}(p,x) \land \text{point}(p) \supset (\exists y) \text{ tin}(x,y,p)\]

**Proposition 4.** \textit{T\textsubscript{wpl\_bundle}} is consistent.

*Proof.* A model constructed by Mace4 can be found at [colore.oor.net/incidence_bundle/consistency/output/wpl_bundle.model](http://colore.oor.net/incidence_bundle/consistency/output/wpl_bundle.model)

**Proposition 5.**
\[ T_{\text{wpl\_bundle}} \models T_{\text{partial\_weak\_point\_line\_existence}} \]

*Proof.* The proofs generated by Prover9 can be found at [colore.oor.net/incidence_bundle/entailment/wpl_bundle\_partial\_weak\_point\_line\_existence/output/](http://colore.oor.net/incidence_bundle/entailment/wpl_bundle\_partial\_weak\_point\_line\_existence/output/)

2.7.2. Introducing \textit{M\textsubscript{wpl\_bundle}}.

**Definition 12.** \( M \in M_{\text{wpl\_bundle}} \) iff
\[ M = (P,L,Q,\text{point}, \text{line}, \text{plane}, \text{in}, \text{tin}) \]

1. \( I = (P,L,Q,\text{point}, \text{line}, \text{plane}, \text{in}) \in M_{\text{partial\_weak\_point\_line\_existence}} \)
2. \( \langle J, \pi, I^* \rangle \) is a \textit{T\textsubscript{partial\_bipartite\_incidence}} bundle such that
\[ \langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p) \]

The next result characterizes the structures within \( M_{\text{wpl\_bundle}} \) up to isomorphism, by using the near-flag complexes of tripartite incidence structures in \( M_{\text{partial\_weak\_point\_line\_existence}} \).

**Theorem 9.** Suppose \( I \in M_{\text{partial\_weak\_point\_line\_existence}} \).
\( \langle F, \pi, I \rangle \) is a \textit{T\textsubscript{partial\_bipartite\_incidence}} bundle iff \( F \) is a near-flag complex for \( I \).

*Proof.* \( \Leftarrow \):
Suppose \( I \in M_{\text{partial\_weak\_point\_line\_existence}} \) and \( F \) is a near-flag complex for \( I \).
If \( (p, l, q) \in F \), then
\[ (p, l), (p, q) \in \text{in}, q \in Q, l \in L, p \in P \]
Thus
\[ q \in N(p) \cap Q \]
\[ l \in N(p) \cap L \]
Let
\[ \pi^{-1}(p) = I_p = \{ (l, q) : (p, l, q) \in F \} \]
Consider the bijection
\[ \varphi : (N(q) \cap L, N(p) \cap Q, I_k) \to \langle X, Y, \text{in}^M \rangle \]
such that
\[ \langle l, q \rangle \in I_p \iff \langle \varphi(l), \varphi(q) \rangle \in \text{in}^M \]
Since
\[ N(p) \cap L = \emptyset, N(p) \cap Q = \emptyset \]
\[ I_p \subseteq N(p) \cap Q \times N(p) \cap L \]
we can see that \( \langle N(p) \cap L, N(p) \cap Q, I_p \rangle \) is definably equivalent to \( \mathcal{M} \) and hence \( \langle F, \pi, I \rangle \) is a \( T_{\text{weak bipartite}} \)-incidence bundle.

Now by the definition of \( \mathfrak{M}_{\text{partial weak point line existence}} \), for any \( q \in Q \), there exists \( p \in P, l \in L \) such that
\[ \langle p, l, q \rangle \in F \]
\[ \langle l, q \rangle \in I_p \]
and hence \( N(q) \cap L \neq \emptyset \).

By the definition of the bijection, if \( \varphi(l) \in \varphi(N(\varphi(q)) \cap X) \) then \( \varphi(q) \in Y \) and
\[ \langle \varphi(l), \varphi(q) \rangle \in \text{in}^M \]
and this is equivalent to \( N(\varphi(q)) \cap X \neq \emptyset \), so that \( \mathcal{M} \) satisfies the definition of \( \mathfrak{M}_{\text{partial bipartite}} \).
\[ \langle N(p) \cap L, N(p) \cap Q, \pi \rangle \text{ is a } T_{\text{partial bipartite}} \text{-incidence bundle.} \]

\[ \Rightarrow \:
\]
Suppose \( \mathcal{I} \in \mathfrak{M}_{\text{weak tripartite}} \) and \( \langle F, \pi, I \rangle \) is a \( T_{\text{weak bipartite}} \)-incidence bundle.
\[ (x, y, z) \in F \text{ iff } \langle z, x \rangle, \langle z, y \rangle \in \pi^{-1}(z) \]
By the definition of incidence bundle,
\[ x \in N(z) \cap Q \]
\[ y \in N(z) \cap L \]
so that
\[ \langle z, x \rangle, \langle z, y \rangle \in \text{in} \]
and hence \( F \) is a near-flag complex. \( \square \)

**Lemma 3.**
\[ \mathfrak{M}_{\text{wp bundle}} \subseteq \mathfrak{M}_{\text{in bundle}} \]

**Proof.** Suppose \( \mathcal{M} = (P, L, Q, \text{point, line, plane, in, tin}) \in \mathfrak{M}_{\text{wp bundle}} \). Since
\[ \mathfrak{M}_{\text{weak point line existence}} \subseteq \mathfrak{M}_{\text{weak tripartite}} \]
\[ \mathcal{I} = (P, L, Q, \text{point, line, plane, in}) \in \mathfrak{M}_{\text{weak tripartite}} \]
\[ \langle J, \pi, I \rangle \text{ is a } T_{\text{partial bipartite}} \text{-incidence bundle iff } \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle \text{ is definably equivalent to some } \mathcal{N} \in \text{Mod}(T_{\text{partial bipartite}}) \text{ for each } p \in P. \]
Since \( \text{Mod}(T_{\text{partial bipartite}}) \subseteq \text{Mod}(T_{\text{weak bipartite}}) \), \( \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle \) is definably equivalent to some \( \mathcal{N} \in \text{Mod}(T_{\text{weak bipartite}}) \) for each \( p \in P \), and
\[ \langle J, \pi, I \rangle \text{ is a } T_{\text{partial bipartite}} \text{-incidence bundle.} \]
By Definition \( \mathcal{M} \in \mathfrak{M}_{\text{in bundle}}. \) \( \square \)
2.7.3. Representation Theorem for $\text{Mod}(T_{\text{wpl bundle}})$.

**Theorem 10.** $\mathcal{M} \in \mathfrak{M}_{\text{wpl bundle}}$ iff $\mathcal{M} \in \text{Mod}(T_{\text{wpl bundle}})$

**Proof.** $\Rightarrow$)

Suppose $\mathcal{M} \in \text{Mod}(T_{\text{wpl bundle}})$ but $\mathcal{M} \notin \mathfrak{M}_{\text{wpl bundle}}$.

Since $T_{\text{wpl bundle}} \models T_{\text{in bundle}}$

by Theorems 4 we know $\mathcal{M} \in \mathfrak{M}_{\text{in bundle}}$.

By Proposition 5, $\mathcal{M}$ satisfies condition (1) in the definition of $\mathfrak{M}_{\text{wpl bundle}}$.

It must therefore be the case that $\langle Q, \pi, I \rangle$ is not a $T_{\text{partial bipartite}}$-incidence bundle.

Since $\mathcal{M} \in \mathfrak{M}_{\text{in bundle}}$, $\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle$ is definably equivalent to some $\mathcal{N} \in \mathfrak{M}_{\text{weak bipartite}}$ for each $p \in P$.

Therefore, there must exist

$q \in N(p) \cap Q, l \in N(p) \cap L$

so that

$\mathcal{M} \models (\exists x,y,p) \text{plane}(x) \land \text{in}(p,x) \land \text{point}(p) \land \neg \text{tin}(x,y,p)$

and $\mathcal{M} \notin \text{Mod}(T_{\text{wpl bundle}})$.

$\Leftarrow$)

Suppose $\mathcal{M} \in \mathfrak{M}_{\text{wpl bundle}}$.

By Lemma 3

$\mathcal{M} \models T_{\text{in bundle}}$

By condition (2) in the definition of $\mathfrak{M}_{\text{wpl bundle}}$, $\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle$ is definably equivalent to some $\mathcal{N} \in \text{Mod}(T_{\text{partial bipartite}})$ for each $p \in P$.

By the representation theorem for $T_{\text{partial bipartite}}$, for every $q \in N(p) \cap Q$ there exists $l \in N(p) \cap L$ such that $\langle l, q \rangle \in \pi^{-1}(p)$.

Now $q \in N(p) \cap Q$ iff

$\langle q \rangle \in \text{plane}, \langle p \rangle \in \text{point}$

$l \in N(p) \cap L$ iff

$\langle l \rangle \in \text{line}, \langle p \rangle \in \text{point}$

$\langle l, q \rangle \in \pi^{-1}(p)$ iff

$\langle q, l, p \rangle \in \text{tin}$

Thus,

$\mathcal{M} \models (\forall x,p) \text{plane}(x) \land \text{in}(p,x) \land \text{point}(p) \supset (\exists y) \text{tin}(x,y,p)$

□

**Corollary 3.** $T_{\text{wpl bundle}}$ is a conservative extension of $T_{\text{weak point line existence}}$.

**Proof.** □

2.8. $T_{\text{nip bundle}}$. 
2.8.1. Axiomatization of $T_{\text{nip bundle}}$.

**Definition 13.** $T_{\text{nip bundle}}$ is the set of axioms in $T_{\text{present bundle}}$ together with

\[(\forall x, p) \text{ line}(x) \land \text{in}(p, x) \land \text{point}(p) \supset (\exists y) \text{tin}(y, x, p)\]

**Proposition 6.** $T_{\text{nip bundle}}$ is consistent.

**Proof.** A model constructed by Mace4 can be found at colore.oor.net/incidence_bundle/consistency/output/nip_bundle.model □

**Proposition 7.**

$T_{\text{nip bundle}} \models T_{\text{partial weak line plane existence}}$

**Proof.** The proofs generated by Prover9 can be found at colore.oor.net/incidence_bundle/entailment/nip_bundle\_partial\_weak\_line\_plane\_existence/output/ □

2.8.2. Introducing $M_{\text{nip bundle}}$.

**Definition 14.** $M \in M_{\text{nip bundle}}$ iff

$M = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle$ such that

1. $I = \langle P, L, Q, \text{point}, \text{line}, \text{plane} \rangle \in M_{\text{partial weak line plane existence}}$;
2. $\langle I, \pi, I^* \rangle$ is a $T_{\text{point bipartite-incidence bundle}}$ such that

$\langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p)$

**Theorem 11.** Suppose $I \in M_{\text{partial weak line plane existence}}$, $\langle F, \pi, I \rangle$ is a $T_{\text{point bipartite-incidence bundle}}$ iff $F$ is a near-flag complex for $I$.

**Proof.** $\Leftarrow$:

Suppose $I \in M_{\text{partial weak point line existence}}$ and $F$ is a near-flag complex for $I$.

If $\langle p, l, q \rangle \in F$, then

$\langle p, l, q \rangle \in \text{in}, q \in Q, l \in L, p \in P$

Thus

$q \in N(p) \cap Q$

$l \in N(p) \cap L$

Let

$\pi^{-1}(p) = I_p = \{ \langle l, q \rangle : \langle p, l, q \rangle \in F \}$

Consider the bijection

$\varphi : \langle N(q) \cap L, N(p) \cap Q, I_p \rangle \to \langle X, Y, \text{in}^M \}$

such that

$\langle l, q \rangle \in I_p \iff \langle \varphi(l), \varphi(q) \rangle \in \text{in}^M$

Since

$N(p) \cap L = \emptyset, N(p) \cap Q = \emptyset$

$I_p \subseteq N(p) \cap Q \times N(p) \cap L$

we can see that $\langle N(p) \cap L, N(p) \cap Q, I_p \rangle$ is definably equivalent to $M$ and hence $\langle F, \pi, I \rangle$ is a $T_{\text{weak bipartite-incidence bundle}}$. 
Now by the definition of \( M_{\text{partial\_weak\_line\_plane\_existence}} \), for any \( l \in L \), there exists \( p \in P, q \in Q \) such that
\[
\langle p, l, q \rangle \in F
\]
\[
\langle l, q \rangle \in I_p
\]
and hence \( N(l) \cap Q \neq \emptyset \).

By the definition of the bijection, if \( \varphi(q) \in \varphi(N(\varphi(l))) \cap Y \) then \( \varphi(l) \in X \) and
\[
\langle \varphi(l), \varphi(q) \rangle \in \text{in}^M
\]
and this is equivalent to \( N(\varphi(l)) \cap Y \neq \emptyset \), so that \( M \) satisfies the definition of \( M_{\text{point\_bipartite}} \).

\[
\langle N(p) \cap L, N(p) \cap Q, I_p \rangle
\]
is definably equivalent to \( M \) and hence \( \langle F, \pi, I \rangle \) is a \( T_{\text{point\_bipartite}} \)-incidence bundle.

\( \Rightarrow \):
Suppose \( I \in M_{\text{weak\_tripartite}} \) and \( \langle F, \pi, I \rangle \) is a \( T_{\text{weak\_bipartite}} \)-incidence bundle.\( (x, y, z) \in F \) iff
\[
\langle z, x \rangle, \langle z, y \rangle \in \pi^{-1}(z)
\]
By the definition of incidence bundle,
\[
x \in N(z) \cap Q
\]
\[
y \in N(z) \cap L
\]
so that
\[
\langle z, x \rangle, \langle z, y \rangle \in \text{in}
\]
and hence \( F \) is a near-flag complex. \( \Box \)

**Lemma 4.**
\[
M_{\text{nip\_bundle}} \subseteq M_{\text{in\_bundle}}
\]

**Proof.** Suppose \( M = (P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin}) \in M_{\text{nip\_bundle}}. \) Since
\[
M_{\text{partial\_weak\_line\_plane\_existence}} \subseteq M_{\text{weak\_tripartite}}
\]
\[
I = (P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}) \in M_{\text{weak\_tripartite}}
\]
\[
\langle J, \pi, I^* \rangle \text{ is a } T_{\text{point\_bipartite}} \text{-incidence bundle iff } \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle
\]
is definably equivalent to some \( N \in \text{Mod}(T_{\text{point\_bipartite}}) \) for each \( p \in P \).

Since \( \text{Mod}(T_{\text{point\_bipartite}}) \subseteq \text{Mod}(T_{\text{weak\_bipartite}}), \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle \)
is definably equivalent to some \( N \in \text{Mod}(T_{\text{weak\_bipartite}}) \) for each \( p \in P \), and
\[
\langle J, \pi, I^* \rangle \text{ is a } T_{\text{point\_bipartite}} \text{-incidence bundle.}
\]

By Definition 14, \( M \in M_{\text{in\_bundle}} \). \( \Box \)

**2.8.3. Representation Theorem for Mod(T_{\text{nip\_bundle}}).**

**Theorem 12.** \( M \in M_{\text{nip\_bundle}} \) iff \( M \in \text{Mod}(T_{\text{nip\_bundle}}) \).

**Proof.** \( \Rightarrow \):
Suppose \( M \in \text{Mod}(T_{\text{nip\_bundle}}) \) but \( M \notin M_{\text{nip\_bundle}} \).

Since
\[
T_{\text{nip\_bundle}} \models T_{\text{in\_bundle}}
\]
by Theorems 1 we know \( M \in M_{\text{in\_bundle}} \).

By Proposition 7, \( M \) satisfies condition (1) in the definition of \( M_{\text{nip\_bundle}} \).
It must therefore be the case that $(Q, \pi, I)$ is not a $T_{\text{point-bipartite}}$-incidence bundle.

Since $M \in \mathfrak{M}_{\text{in-bundle}}$, $(N(p) \cap L, N(p) \cap Q, \pi^{-1}(p))$ is definably equivalent to some $N' \in \mathfrak{M}_{\text{weak-bipartite}}$ for each $p \in P$.

Therefore, there must exist
\[ q \in N(p) \cap Q, l \in N(p) \cap L, \langle l, q \rangle \notin \pi^{-1}(p) \]
so that
\[ M \models (\exists x, y, p) \text{line}(x) \land \text{in}(p, x) \land \text{point}(p) \land \neg \text{tin}(x, y, p) \]
and $M \not\in \text{Mod}(T_{\text{in-bundle}})$.

Suppose $M \in \mathfrak{M}_{\text{wp-bundle}}$.

By Lemma 1
\[ M \models T_{\text{in-bundle}} \]
By condition (2) in the definition of $\mathfrak{M}_{\text{nip-bundle}}$, $(N(p) \cap L, N(p) \cap Q, \pi^{-1}(p))$ is definably equivalent to some $N \in \text{Mod}(T_{\text{point-bipartite}})$ for each $p \in P$.

By the representation theorem for $T_{\text{point-bipartite}}$, for every $l \in N(p) \cap L$ there exists $q \in N(p) \cap Q$ such that $\langle l, q \rangle \in \pi^{-1}(p)$.

Now $q \in N(p) \cap Q$ iff $\langle q \rangle \in \text{plane}$, $\langle p \rangle \in \text{point}$
\[ l \in N(p) \cap L \text{ iff } \langle l \rangle \in \text{line}, \langle p \rangle \in \text{point} \]
\[ \langle l, q \rangle \in \pi^{-1}(p) \text{ iff } \langle q, l, p \rangle \in \text{tin} \]
Thus,
\[ M \models (\forall x, p) \text{line}(x) \land \text{in}(p, x) \land \text{point}(p) \lor (\exists y) \text{tin}(x, y, p) \]

Corollary 4. $T_{\text{nip-bundle}}$ is a conservative extension of $T_{\text{weak-line-plane-existence}}$.

Proof. □

2.9. $T_{lp}$-bundle.

2.9.1. Axiomatization of $T_{lp}$-bundle.

Definition 15. $T_{lp}$-bundle is the set of axioms in $T_{\text{in-bundle}}$ together with
\[
(\forall x) \text{line}(x) \lor (\exists y, p) \text{tin}(y, x, p)
\]

Proposition 8. $T_{lp}$-bundle is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/incidence_bundle/consistency/output/lp_bundle.model □

Proposition 9.
\[ T_{lp} \models T_{\text{partial-line-near-flag}} \]

Proof. The proofs generated by Prover9 can be found at colore.oor.net/incidence_bundle/entailment/lp_bundleFpartial_line_near_flag/output/ □
2.9.2. Introducing $\mathcal{M}_{lp\text{-}bundle}$.

**Definition 16.** $\mathcal{M} \in \mathcal{M}_{lp\text{-}bundle}$ iff

$\mathcal{M} = \langle P, L, Q, \text{point, line, plane, in, tin} \rangle$ such that

1. $I = \langle P, L, Q, \text{point, line, plane, in} \rangle \in \mathcal{M}_{\text{partial\_line\_near\_flag}}$,
2. $\langle J, \pi, I^* \rangle$ is a $\mathcal{T}_\text{point\_bipartite\_incidence}$ bundle such that
   \[ \langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p) \]

**Lemma 5.**

$\mathcal{M}_{lp\text{-}bundle} \subseteq \mathcal{M}_{nip\text{-}bundle} \subseteq \mathcal{M}_{in\text{-}bundle}$

**Proof.** Suppose $\mathcal{M} = \langle P, L, Q, \text{point, line, plane, in, tin} \rangle \in \mathcal{M}_{nip\text{-}bundle}$. Since $\mathcal{M}_{\text{partial\_line\_near\_flag}} \subseteq \mathcal{M}_{\text{partial\_weak\_line\_plane\_existence}}$, $I = \langle P, L, Q, \text{point, line, plane, in} \rangle \in \mathcal{M}_{\text{partial\_weak\_line\_plane\_existence}}$.

$\langle J, \pi, I \rangle$ is a $\mathcal{T}_\text{point\_bipartite\_incidence}$ bundle iff $\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle$ is definably equivalent to some $N \in \text{Mod}(\mathcal{T}_\text{point\_bipartite})$ for each $p \in P$.

Since $\text{Mod}(\mathcal{T}_\text{point\_bipartite}) \subseteq \text{Mod}(\mathcal{T}_\text{weak\_bipartite})$, $\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle$ is definably equivalent to some $N \in \text{Mod}(\mathcal{T}_\text{weak\_bipartite})$ for each $p \in P$, and $\langle J, \pi, I \rangle$ is a $\mathcal{T}_\text{point\_bipartite\_incidence}$ bundle.

By Definition 14, $\mathcal{M} \in \mathcal{M}_{\text{nip\_bundle}}$.

**Theorem 13.** Suppose $I \in \mathcal{M}_{\text{partial\_line\_near\_flag}}$.

$\langle F, \pi, I \rangle$ is a $\mathcal{T}_\text{point\_bipartite\_incidence}$ bundle iff $F$ is a near-flag complex for $I$.

**Proof.** This follows from Theorem 11 and Lemma 5.

**2.9.3. Representation Theorem for $\text{Mod}(\mathcal{T}_{lp\text{-}bundle})$.**

**Theorem 14.** $\mathcal{M} \in \mathcal{M}_{lp\text{-}bundle}$ iff $\mathcal{M} \in \text{Mod}(\mathcal{T}_{lp\text{-}bundle})$.

**Proof.** $\Rightarrow$:

Suppose $\mathcal{M} \in \text{Mod}(\mathcal{T}_{lp\text{-}bundle})$ but $\mathcal{M} \notin \mathcal{M}_{lp\text{-}bundle}$.

Since $\mathcal{T}_{lp\text{-}bundle} \vDash \mathcal{T}_{in\text{-}bundle}$ by Theorems 4, we know $\mathcal{M} \in \mathcal{M}_{in\text{-}bundle}$.

By Proposition 9, $\mathcal{M}$ satisfies condition (1) in the definition of $\mathcal{M}_{lp\text{-}bundle}$.

It must therefore be the case that $\langle Q, \pi, I \rangle$ is not a $\mathcal{T}_\text{partial\_bipartite\_incidence}$ bundle.

Since $\mathcal{M} \in \mathcal{M}_{in\text{-}bundle}$, $\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle$ is definably equivalent to some $N^* \in \mathcal{M}_{\text{weak\_bipartite}}$ for each $p \in P$.

Therefore, there must exist

$q \in N(p) \cap Q, l \in N(p) \cap L$

$\langle l, q \rangle \notin \pi^{-1}(p)$

so that

$\mathcal{M} \models (\exists x, y, p) \ line(x) \land \ tin(x, y, p)$

and $\mathcal{M} \notin \text{Mod}(\mathcal{T}_{lp\text{-}bundle})$.

$\Leftarrow$:

Suppose $\mathcal{M} \in \mathcal{M}_{lp\text{-}bundle}$.
By Lemma 5

\[ \mathcal{M} \models T_{\text{in\_bundle}} \]

By condition (2) in the definition of \( \mathfrak{M}^{\text{lp\_bundle}} \), \( (N(p) \cap L, N(p) \cap Q, \pi^{-1}(p)) \) is definably equivalent to some \( \mathcal{N} \in \text{Mod}(T_{\text{point\_bipartite}}) \) for each \( p \in P \).

By the representation theorem for \( T_{\text{point\_bipartite}} \), for every \( l \in N(p) \cap L \) there exists \( q \in N(p) \cap Q \) such that \( \langle l, q \rangle \in \pi^{-1}(p) \).

Now \( q \in N(p) \cap Q \) iff

\[ \langle q \rangle \in \text{plane}, \langle p \rangle \in \text{point} \]

\( l \in N(p) \cap L \) iff

\[ \langle l \rangle \in \text{line}, \langle p \rangle \in \text{point} \]

\( \langle l, q \rangle \in \pi^{-1}(p) \) iff

\[ \langle q, l, p \rangle \in \text{tin} \]

Thus,

\[ \mathcal{M} \models (\forall x, p) \text{line}(x) \land \text{in}(p, x) \land \text{point}(p) \supset (\exists y) \text{tin}(x, y, p) \]

Furthermore, since \( I \in (\mathfrak{M}^{\text{partial\_tripartite}}) \),

\[ \mathcal{M} \models (\forall x) \text{line}(x) \supset (\exists p) \text{in}(p, x) \land \text{point}(p) \]

so that

\[ \mathcal{M} \models (\forall x) \text{line}(x) \supset (\exists y) \text{tin}(x, y, p) \]

\[ \square \]

Corollary 5. \( T_{\text{lp\_bundle}} \) is a conservative extension of \( T_{\text{partial\_plane\_near\_flag}} \).

Proof.

\[ \square \]

2.10. \( T_{\text{pp\_bundle}} \).

2.10.1. Axiomatization of \( T_{\text{pp\_bundle}} \).

Definition 17. \( T_{\text{pp\_bundle}} \) is the set of axioms in \( T_{\text{present\_bundle}} \) together with

(14)

\[ (\forall x) \text{plane}(x) \supset (\exists y, p) \text{tin}(x, y, p) \]

Proposition 10. \( T_{\text{pp\_bundle}} \) is consistent.

Proof. A model constructed by Mace4 can be found at \texttt{colo.re.oor.net/incidence_bundle/consistency/output/pp_bundle.model} \[ \square \]

Proposition 11.

\[ T_{\text{pp\_bundle}} \models T_{\text{partial\_plane\_near\_flag}} \]

Proof. The proofs generated by Prover9 can be found at \texttt{colo.re.oor.net/incidence_bundle/entailment/pp_bundleFpartial_plane_near_flag/output/} \[ \square \]
2.10.2. **Introducing \( \mathcal{M}^{pp\text{-bundle}} \).**

**Definition 18.** \( \mathcal{M} \in \mathcal{M}^{pp\text{-bundle}} \) iff

\[ \mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \] such that

1. \( I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \rangle \in \mathcal{M}^{\text{partial plane near flag}} \);
2. \( \langle J, \pi, I^* \rangle \) is a \( T^{\text{partial bipartite-incidence bundle}} \) such that
   \[ (q, l, p) \in \text{tin} \iff (q, l) \in \pi^{-1}(p) \]

**Lemma 6.**

\[ \mathcal{M}^{pp\text{-bundle}} \subseteq \mathcal{M}^{wpl\text{-bundle}} \subseteq \mathcal{M}^{\text{in bundle}} \]

**Proof.**

Suppose \( \mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathcal{M}^{pp\text{-bundle}} \). Since

\[ \mathcal{M}^{\text{partial plane near flag}} \subseteq \mathcal{M}^{\text{partial weak point line existence}} \]

\[ \langle J, \pi, I^* \rangle \) is a \( T^{\text{partial bipartite-incidence bundle}} \) iff \( (N(p) \cap L, N(p) \cap Q, \pi^{-1}(p)) \) is definably equivalent to some \( N \in \text{Mod}(T^{\text{partial bipartite}}) \) for each \( p \in P \).

Since \( \text{Mod}(T^{\text{partial bipartite}}) \subseteq \text{Mod}(T^{\text{weak bipartite}}) \), \( (N(p) \cap L, N(p) \cap Q, \pi^{-1}(p)) \) is definably equivalent to some \( N \in \text{Mod}(T^{\text{weak bipartite}}) \) for each \( p \in P \), and

\[ \langle J, \pi, I^* \rangle \) is a \( T^{\text{partial bipartite-incidence bundle}} \).

By Definition 18, \( \mathcal{M} \in \mathcal{M}^{wpl\text{-bundle}} \).

**Theorem 15.** Suppose \( \mathcal{I} \in \mathcal{M}^{\text{partial plane near flag}} \).

\[ \langle F, \pi, \mathcal{I} \rangle \) is a \( T^{\text{partial bipartite-incidence bundle}} \) iff \( F \) is a near-flag complex for \( I \).

**Proof.** This follows from Theorem 9 and Lemma 9.

2.10.3. **Representation Theorem for \( \text{Mod}(T^{pp\text{-bundle}}) \).**

**Theorem 16.** \( \mathcal{M} \in \mathcal{M}^{pp\text{-bundle}} \) iff \( \mathcal{M} \in \text{Mod}(T^{pp\text{-bundle}}) \).

**Proof.** \( \Rightarrow \):

Suppose \( \mathcal{M} \in \text{Mod}(T^{pp\text{-bundle}}) \) but \( \mathcal{M} \notin \mathcal{M}^{pp\text{-bundle}} \).

Since

\[ T^{pp\text{-bundle}} \models T^{\text{in bundle}} \]

by Theorems 4, we know \( \mathcal{M} \in \mathcal{M}^{\text{in bundle}} \).

By Proposition 11, \( \mathcal{M} \) satisfies condition (1) in the definition of \( \mathcal{M}^{pp\text{-bundle}} \).

It must therefore be the case that \( \langle Q, \pi, I \rangle \) is not a \( T^{\text{point bipartite-incidence bundle}} \).

Since \( \mathcal{M} \in \mathcal{M}^{\text{in bundle}} \), \( (N(p) \cap L, N(p) \cap Q, \pi^{-1}(p)) \) is definably equivalent to some \( N' \in \text{Mod}(T^{\text{weak bipartite}}) \) for each \( p \in P \).

Therefore, there must exist

\[ q \in N(p) \cap Q, l \in N(p) \cap L \]

\[ (l, q) \notin \pi^{-1}(p) \]

so that

\[ \mathcal{M} \models (\exists x, y, p) \text{ line}(x) \land \neg \text{tin}(x, y, p) \]

and \( \mathcal{M} \notin \text{Mod}(T^{\text{nip bundle}}) \).

\( \Leftarrow \):

Suppose \( \mathcal{M} \in \mathcal{M}^{pp\text{-bundle}} \).
By Lemma 6

\[ \mathcal{M} \models T_{in\text{-bundle}} \]

By condition (2) in the definition of \( T_{pp\text{-bundle}} \), \( \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle \) is definably equivalent to some \( N \in \text{Mod}(T_{\text{partial\_bipartite}}) \) for each \( p \in P \).

By the representation theorem for \( T_{\text{partial\_bipartite}} \), for every \( q \in N(p) \cap Q \) there exists \( l \in N(p) \cap L \) such that \( \langle l, q \rangle \in \pi^{-1}(p) \).

Now \( q \in N(p) \cap Q \) iff

\[ \langle q \rangle \in \text{plane}, \langle p \rangle \in \text{point} \]

\( l \in N(p) \cap L \) iff

\[ \langle l \rangle \in \text{line}, \langle p \rangle \in \text{point} \]

\( \langle l, q \rangle \in \pi^{-1}(p) \) iff

\[ \langle q, l, p \rangle \in \text{tin} \]

Thus,

\[ \mathcal{M} \models (\forall x, p) \text{plane}(x) \land \text{in}(p, x) \land \text{point}(p) \supset (\exists y) \text{tin}(x, y, p) \]

Furthermore, since \( I \in (T_{\text{point\_plane\_existence}}) \),

\[ \mathcal{M} \models (\forall x) \text{plane}(x) \supset (\exists p) \text{in}(p, x) \land \text{point}(p) \]

so that

\[ \mathcal{M} \models (\forall x) \text{plane}(x) \supset (\exists y) \text{tin}(x, y, p) \]

\[ \square \]

Corollary 6. \( T_{pp\text{-bundle}} \) is a conservative extension of \( T_{\text{point\_plane\_existence}} \).

Proof. \[ \square \]

2.11. \( T_{\text{plane\_flag\_bundle}} \).

2.11.1. Axiomatization of \( T_{\text{plane\_flag\_bundle}} \).

Definition 19.

\[ T_{\text{plane\_flag\_bundle}} = T_{\text{nip\_bundle}} \cup T_{pp\text{-bundle}} \]

Proposition 12. \( T_{\text{plane\_flag\_bundle}} \) is consistent.

Proof. A model constructed by Mace4 can be found at [colore.oor.net/incidence_bundle/consistency/output/plane_flag_bundle.model](http://colore.oor.net/incidence_bundle/consistency/output/plane_flag_bundle.model) \[ \square \]

Proposition 13.

\[ T_{\text{plane\_flag\_bundle}} \models T_{\text{partial\_strong\_plane\_near\_flag}} \]

Proof. The proofs generated by Prover9 can be found at [colore.oor.net/incidence_bundle/entailment/plane_flag_bundleFpartial_strong_plane_near_flag/output/](http://colore.oor.net/incidence_bundle/entailment/plane_flag_bundleFpartial_strong_plane_near_flag/output/) \[ \square \]
2.11.2. **Introducing** $\mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle}$.  

**Definition 20.** \( \mathcal{M} \in \mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle} \) iff  
\[ \mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \] such that  
(1) \( I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathcal{M}_\text{partial_strong_line_near_flag}^\text{partial_strong_line_near_flag} \),  
(2) \( \langle J, \pi, I^* \rangle \) is a \( T_{\text{bipartite}} \)-incidence bundle such that  
\[ \langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p) \]

The incidence bundle in Figure 1 depicts a structure in \( \text{Mod}(T_{\text{flag_bundle}}^\text{flag_bundle}) \). Each line and plane is incident with a point within the tripartite incidence structure. Each line is an element of a fibre in the incidence bundle, and each plane is an element of a fibre.

**Lemma 7.**  
\[ \mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle} = \mathcal{M}_\text{nip_bundle}^\text{nip_bundle} \cap \mathcal{M}_\text{pp_bundle}^\text{pp_bundle} \]

**Proof.** Suppose \( \mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle} \). Since \( \mathcal{M}_\text{partial_strong_line_near_flag}^\text{partial_strong_line_near_flag} \) we can see that  
\[ I = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathcal{M}_\text{partial_strong_line_near_flag}^\text{partial_strong_line_near_flag} \]

\[ \langle J, \pi, I^* \rangle \text{ is a } T_{\text{bipartite}} \text{-incidence bundle iff } \langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle \text{ is definably equivalent to some } N \in \text{Mod}(T_{\text{bipartite}}^\text{bipartite}) \text{ for each } p \in P. \]

Now \( N \in \text{Mod}(T_{\text{bipartite}}^\text{bipartite}) \) iff \( N \in \text{Mod}(T_{\text{partial_bipartite}}^\text{partial_bipartite} \cup T_{\text{point_bipartite}}^\text{point_bipartite}) \) iff \( N \in \text{Mod}(T_{\text{partial_bipartite}}^\text{partial_bipartite}) \cap \text{Mod}(T_{\text{point_bipartite}}^\text{point_bipartite}) \), so \( \langle J, \pi, I^* \rangle \) is both a \( T_{\text{partial_bipartite}} \)-incidence bundle and a \( T_{\text{point_bipartite}} \)-incidence bundle.

By Definition 20, \( \mathcal{M} \in \mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle} \). \( \square \)

**Theorem 17.** Suppose \( \mathcal{I} \in \mathcal{M}_\text{partial_strong_line_near_flag}^\text{partial_strong_line_near_flag} \). \( \langle F, \pi, \mathcal{I} \rangle \) is a \( T_{\text{bipartite}} \)-incidence bundle iff \( \mathcal{F} \) is a near-flag complex for \( \mathcal{I} \).

**Proof.** This follows from Theorem 11, Theorem 15, and Lemma 7. \( \square \)

2.11.3. **Representation Theorem for** \( \text{Mod}(T_{\text{plane_flag_bundle}}^\text{plane_flag_bundle}) \).

**Theorem 18.** \( \mathcal{M} \in \text{Mod}(T_{\text{plane_flag_bundle}}^\text{plane_flag_bundle}) \) iff \( \mathcal{M} \in \mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle} \).

**Proof.** \( \mathcal{M} \in \text{Mod}(T_{\text{plane_flag_bundle}}^\text{plane_flag_bundle}) \) iff \( \mathcal{M} \in \text{Mod}(T_{\text{nip_bundle}}^\text{nip_bundle} \cup T_{\text{pp_bundle}}^\text{pp_bundle}) \) (by Definition) \( \cap \mathcal{M} \in \text{Mod}(T_{\text{nip_bundle}}^\text{nip_bundle} \cap T_{\text{pp_bundle}}^\text{pp_bundle}) \) (by Theorems 12 and 16) \( \mathcal{M} \in \mathcal{M}_\text{plane_flag_bundle}^\text{plane_flag_bundle} \) (by Lemma 7). \( \square \)

**Corollary 7.** \( T_{\text{plane_flag_bundle}}^\text{plane_flag_bundle} \) is a conservative extension of \( T_{\text{line_plane_existence}}^\text{line_plane_existence} \).

**Proof.**

2.12. **\( T_{\text{line_flag_bundle}}^\text{line_flag_bundle} \).**
2.12.1. Axiomatization of $T_{\text{line flag bundle}}$.

Definition 21.

$$T_{\text{line flag bundle}} = T_{\text{wpb bundle}} \cup T_{\text{lp bundle}}$$

Proposition 14. $T_{\text{line flag bundle}}$ is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/incidence_bundle/consistency/output/line_flag_bundle.model

Proposition 15.

$$T_{\text{line flag bundle}} \models T_{\text{partial strong line near flag}}$$

Proof. The proofs generated by Prover9 can be found at colore.oor.net/incidence_bundle/entailment/line_flag_bundle\_partial_strong_line_near_flag/output/□

2.12.2. Introducing $M_{\text{line flag bundle}}$.

Definition 22. $M \in M_{\text{line flag bundle}}$ iff

$$M = \langle P, L, Q, \text{point, line, plane, in, tin} \rangle$$

such that

1. $I = \langle P, L, Q, \text{point, line, plane, in} \rangle \in M_{\text{partial strong line near flag}}$;
2. $\langle J, \pi, I^* \rangle$ is a $T_{\text{bipartite}}$-incidence bundle such that

$$\langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p)$$

Lemma 8.

$$M_{\text{line flag bundle}} = M_{\text{wpb bundle}} \cap M_{\text{lp bundle}}$$

Proof. Suppose $M = \langle P, L, Q, \text{point, line, plane, in, tin} \rangle \in M_{\text{line flag bundle}}$. Since $M_{\text{partial strong line near flag}} = M_{\text{partial weak point line existence}} \cap M_{\text{partial line near flag}}$ we can see that

$$I = \langle P, L, Q, \text{point, line, plane, in} \rangle \in M_{\text{partial weak point line existence}} \cap M_{\text{partial line near flag}}$$

$\langle J, \pi, I^* \rangle$ is a $T_{\text{bipartite}}$-incidence bundle iff $\langle N(p) \cap L, N(p) \cap Q, \pi^{-1}(p) \rangle$ is definably equivalent to some $N \in \text{Mod}(T_{\text{bipartite}})$ for each $p \in P$.

Now $N \in \text{Mod}(T_{\text{bipartite}})$ iff $N \in \text{Mod}(T_{\text{partial bipartite}} \cup T_{\text{point bipartite}})$ iff $N \in \text{Mod}(T_{\text{partial bipartite}}) \cap \text{Mod}(T_{\text{point bipartite}})$, so $\langle J, \pi, I^* \rangle$ is both a $T_{\text{partial bipartite}}$-incidence bundle and a $T_{\text{point bipartite}}$-incidence bundle.

By Definition 22, $M \in M_{\text{line flag bundle}}$. □

Theorem 19. Suppose $I \in M_{\text{partial strong line near flag}}$

$\langle F, \pi, I \rangle$ is a $T_{\text{bipartite}}$-incidence bundle iff $F$ is a flag complex for $I$.

Proof. This follows from Theorem 9, Theorem 13, and Lemma 8. □
2.12.3. Representation Theorem for \( \text{Mod}(T_{\text{line_flag.bundle}}) \).

**Theorem 20.** \( \mathcal{M} \in \text{Mod}(T_{\text{line_flag.bundle}}) \) iff \( \mathcal{M} \in \mathfrak{M}^{\text{line_flag.bundle}} \).

**Proof.** \( \mathcal{M} \in \text{Mod}(T_{\text{line_flag.bundle}}) \) iff
- \( \mathcal{M} \in \text{Mod}(T_{\text{wpl.bundle}} \cup T_{\text{lp.bundle}}) \) (by Definition)
- \( \mathcal{M} \in \mathfrak{M}^{\text{wpl.bundle}} \cap \mathfrak{M}^{\text{lp.bundle}} \) (by Theorems 10 and 14)
- \( \mathcal{M} \in \mathfrak{M}^{\text{line_flag.bundle}} \) (by Lemma 8).

**Corollary 8.** \( T_{\text{line_flag.bundle}} \) is a conservative extension of \( T_{\text{point_line_existence}} \).

**Proof.**

2.13. \( T_{\text{flag.bundle}} \).

2.13.1. Axiomatization of \( T_{\text{flag.bundle}} \).

**Definition 23.**

\( T_{\text{flag.bundle}} = T_{\text{line_flag.bundle}} \cup T_{\text{plane_flag.bundle}} \)

**Proposition 16.** \( T_{\text{flag.bundle}} \) is consistent.

**Proof.** A model constructed by Mace4 can be found at [colore.oor.net/incidence_bundle/consistency/output/flag_bundle.model]

**Proposition 17.** \( T_{\text{flag.bundle}} \models T_{\text{partial_strong_near_flag}} \)

**Proof.** The proofs generated by Prover9 can be found at [colore.oor.net/incidence_bundle/entailment/flag_bundlePartial_strong_near_flag/output/]

2.13.2. Introducing \( \mathfrak{M}^{\text{flag.bundle}} \).

**Definition 24.** \( \mathcal{M} \in \mathfrak{M}^{\text{flag.bundle}} \) iff \( \mathcal{M} = \langle P, L, Q, \text{point, line, plane, in, tin} \rangle \) such that
- \( I = \langle P, L, Q, \text{point, line, plane, in} \rangle \in \mathfrak{M}^{\text{partial_strong_near_flag}}, \)
- \( \langle \Pi, \pi, \Pi' \rangle \) is a \( T_{\text{bipartite}} \)-incidence bundle such that \( \langle q, l, p \rangle \in \text{tin} \iff \langle q, l \rangle \in \pi^{-1}(p) \)

**Lemma 9.**

\( \mathfrak{M}^{\text{flag.bundle}} = \mathfrak{M}^{\text{line_flag.bundle}} \cap \mathfrak{M}^{\text{plane_flag.bundle}} \)

**Proof.** Suppose \( \mathcal{M} = \langle P, L, Q, \text{point, line, plane, in, tin} \rangle \in \mathfrak{M}^{\text{line_flag.bundle}} \). Since \( \mathfrak{M}^{\text{partial_strong_near_flag}} = \mathfrak{M}^{\text{partial_strong_line_near_flag}} \cap \mathfrak{M}^{\text{partial_strong_plane_near_flag}} \) we can see that

\( I = \langle P, L, Q, \text{point, line, plane, in} \rangle \in \mathfrak{M}^{\text{partial_strong_line_near_flag}} \cap \mathfrak{M}^{\text{partial_strong_plane_near_flag}} \)

By Definition 24, \( \mathcal{M} \in \mathfrak{M}^{\text{flag.bundle}} \).

**Theorem 21.** Suppose \( \Pi \in \mathfrak{M}^{\text{partial_strong_near_flag}} \).

\( \langle \mathcal{F}, \pi, \Pi \rangle \) is a \( T_{\text{bipartite}} \)-incidence bundle iff \( \mathcal{F} \) is a near-flag complex for \( \Pi \).

**Proof.** This follows from Theorem 17, Theorem 19 and Lemma 9.
2.13.3. Representation Theorem for $\text{Mod}(T_{\text{flag bundle}})$.

**Theorem 22.** $\mathcal{M} \in \text{Mod}(T_{\text{flag bundle}})$ iff $\mathcal{M} \in \mathfrak{M}_{\text{flag bundle}}$.

**Proof.**

$\mathcal{M} \in \text{Mod}(T_{\text{flag bundle}})$ iff $\mathcal{M} \in \text{Mod}(T_{\text{line-flag bundle}} \cup T_{\text{plane-flag bundle}})$ (by Definition)

iff $\mathcal{M} \in \text{Mod}(T_{\text{line-flag bundle}}) \cap \text{Mod}(T_{\text{plane-flag bundle}})$ (by Theorems 20, 18)

iff $\mathcal{M} \in \mathfrak{M}_{\text{line-flag bundle}} \cap \mathfrak{M}_{\text{plane-flag bundle}}$ (by Lemma 9). □

**Corollary 9.** $T_{\text{flag bundle}}$ is a conservative extension of $T_{\text{flag existence}}$.

**Proof.** □

2.14. **Summary.** We can see from Definition 6 that in any model of an incidence bundle ontology, there are two primary substructures – a “global” tripartite incidence structure and the “local” bipartite incidence structures which are the fibres of the incidence bundle. This is also reflected in the axiomatizations of theories in the $H_{\text{incidence bundle}}$ Hierarchy, whose relationships are shown in Figure 3. Each theory is a conservative extension of a theory of tripartite incidence structures and a nonconservative extension of the root theory for incidence bundles. The relationships between each of the ontologies in the hierarchy and the ontologies which axiomatize these substructures are summarized in Table 1.

### Table 1. Incidence bundle ontologies.

<table>
<thead>
<tr>
<th>Incidence Bundle Ontology</th>
<th>Tripartite Incidence Structure</th>
<th>Fibre Ontology</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{\text{line bundle}}$</td>
<td>$T_{\text{weak-tripartite}}$</td>
<td>$T_{\text{weak-bipartite}}$</td>
</tr>
<tr>
<td>$T_{\text{wpbundle}}$</td>
<td>$T_{\text{partial-weakening-line-existence}}$</td>
<td>$T_{\text{partial-bipartite}}$</td>
</tr>
<tr>
<td>$T_{\text{line bundle}}$</td>
<td>$T_{\text{partial-weakening-line-existence}}$</td>
<td>$T_{\text{point-bipartite}}$</td>
</tr>
<tr>
<td>$T_{\text{plane flag bundle}}$</td>
<td>$T_{\text{partial-strong-line-near-flag}}$</td>
<td>$T_{\text{point-bipartite}}$</td>
</tr>
<tr>
<td>$T_{\text{flag bundle}}$</td>
<td>$T_{\text{partial-strong-line-near-flag}}$</td>
<td>$T_{\text{point-bipartite}}$</td>
</tr>
</tbody>
</table>

3. **Incidence Bundles and Participation Ontologies**

3.1. **Verification of $T_{\text{psl-participates}}$.** $T_{\text{psl-participates}}$ is the set of axioms in Figure 4.

Note that $T_{\text{psl-core}}$ is a conservative extension of $T_{\text{psl-participates}}$, that is, $T_{\text{psl-participates}}$ is a module of $T_{\text{psl-core}}$.

---

7 The axioms for the incidence bundle ontologies can be found at [colore.oor.net/incidence bundles](https://colore.oor.net/incidence_bundles)

The axioms for the tripartite incidence structures can be found at [colore.oor.net/tripartite incidence](https://colore.oor.net/tripartite_incidence)

The axioms for the bipartite incidence structures can be found at [colore.oor.net/bipartite incidence](https://colore.oor.net/bipartite_incidence)
\[(\forall o) \text{ activity occurrence}(o) \supset \neg \text{object}(o) \land \neg \text{timepoint}(o)\]

\[(\forall x) \text{ object}(x) \supset \neg \text{timepoint}(x)\]

\[(\forall o, t) \text{ is occurring at}(o, t) \supset \text{activity occurrence}(o) \land \text{timepoint}(t)\]

\[(\forall x, t) \text{ exists at}(x, t) \supset \text{object}(x) \land \text{timepoint}(t)\]

\[(\forall x, o, t) \text{ participates in}(x, o, t) \supset \text{object}(x) \land \text{activity occurrence}(o) \land \text{timepoint}(t)\]

\[(\forall x, o, t) \text{ participates in}(x, o, t) \supset \text{exists at}(x, t) \land \text{is occurring at}(o, t)\]

Figure 4. \(T_{\text{psl\_participates}}\): Axioms of the module of \(T_{\text{psl\_core}}\).

**Definition 25.** \(\Delta_{\text{psl2inbundle}}\) is the following set of translation definitions:

\[(\forall p) \text{ point}(p) \equiv \text{timepoint}(p)\]

\[(\forall l) \text{ line}(l) \equiv \text{activity occurrence}(l)\]

\[(\forall q) \text{ plane}(q) \equiv \text{object}(q)\]

\[(\forall x, o, t) \text{ tin}(o, x, t) \equiv \text{participates in}(x, o, t)\]

\[(\forall x, y) \text{ in}(x, y) \equiv \text{is occurring at}(y, x) \lor \text{exists at}(y, x)\]

\[\Pi_{\text{inbundle2psl}}\] is the following set of translation definitions:

\[(\forall t) \text{ timepoint}(t) \equiv \text{point}(t)\]

\[(\forall o) \text{ activity occurrence}(o) \equiv \text{line}(o)\]

\[(\forall x) \text{ object}(x) \equiv \text{plane}(x)\]

\[(\forall o, t) \text{ is occurring at}(o, t) \equiv \text{in}(t, o) \land \text{line}(o) \land \text{point}(t)\]

\[(\forall x, t) \text{ exists at}(x, t) \equiv \text{in}(t, x) \land \text{plane}(x) \land \text{point}(t)\]

\[(\forall x, o, t) \text{ participates in}(x, o, t) \equiv \text{tin}(o, x, t)\]

**Proposition 18.**

\[T_{\text{psl\_participates}} \cup \Delta_{\text{psl2inbundle}} \models T_{\text{present\_bundle}}\]

**Proof.** Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/psl2inbundle/output/
Figure 5. $T_{\text{gangemi}}$: Axioms Gangemi’s ontology pattern.

Proposition 19.

$T_{\text{present\_bundle}} \cup \Pi_{\text{inbundle2psl}} \models T_{\text{psl\_participates}}$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/inbundle2psl/output/.

Proposition 20.

$T_{\text{psl\_participates}} \cup \Delta_{\text{psl2inbundle}} \models \Pi_{\text{inbundle2psl}}$

$T_{\text{in\_bundle}} \cup \Pi_{\text{inbundle2psl}} \models \Delta_{\text{psl2inbundle}}$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/psl2inbundle_map/output/ and colore.oor.net/psl_participates/interprets/inbundle2psl_map/output/.

Theorem 23. $T_{\text{psl\_participates}}$ is synonymous with $T_{\text{present\_bundle}}$.

Proof. By Proposition 18, $T_{\text{psl\_participates}}$ interprets $T_{\text{present\_bundle}}$. This interpretation is faithful because $T_{\text{psl\_participates}} \cup \Delta_{\text{psl2inbundle}}$ is a conservative extension of $T_{\text{present\_bundle}}$. By Proposition 19, $T_{\text{present\_bundle}}$ interprets $T_{\text{psl\_participates}}$. This interpretation is faithful because $T_{\text{present\_bundle}} \cup \Pi_{\text{inbundle2psl}}$ is a conservative extension of $T_{\text{psl\_participates}}$. By Proposition 20 and [9], $T_{\text{psl\_participates}}$ is synonymous with $T_{\text{present\_bundle}}$. □

Corollary 10. $\text{Mod}(T_{\text{psl\_participates}})$ is representable by $\text{Mod}(T_{\text{present\_bundle}})$.
3.2. Verification of $T_{\text{gangemi}}$. Next, we consider the first-order axiomatization of an extension of Gangemi’s ontology pattern in Figure 5.

Definition 26. $\Delta_{\text{gangemi2inbundle}}$ is the following set of translation definitions:

$\forall p \, \text{point}(p) \equiv \text{time}(p)$
$\forall l \, \text{line}(l) \equiv \text{event}(l)$
$\forall q \, \text{plane}(q) \equiv \text{object}(q)$

$\forall x, o, t \, \text{tin}(o, x, t) \equiv \text{hasParticipant}(o, x) \land \text{pre}(x, t) \land \text{pre}(o, t)$

$\forall x, y \, \text{in}(x, y) \equiv \text{pre}(x, y)$

$\Pi_{\text{inbundle2gangemi}}$ is the following set of translation definitions:

$\forall t \, \text{time}(t) \equiv \text{point}(t)$
$\forall o \, \text{event}(o) \equiv \text{line}(o)$
$\forall x \, \text{object}(x) \equiv \text{plane}(x)$

$\forall x, t \, \text{pre}(x, y) \equiv \text{in}(x, y)$

$\forall x, o, t \, \text{hasParticipant}(o, x) \equiv (\exists t) \, \text{tin}(o, x, t)$

Proposition 21.

$T_{\text{gangemi}} \cup \Delta_{\text{gangemi2inbundle}} \models T_{\text{present bundle}}$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/gangemi_participates/interprets/gangemi2present_bundle/output/

This shows that $T_{\text{gangemi}}$ interprets $T_{\text{present bundle}}$, but the interpretation is not faithful – there are sentences in the signature of $T_{\text{present bundle}}$ which are entailed by $T_{\text{gangemi}} \cup \Delta$ which are not entailed by $T_{\text{present bundle}}$. We therefore need to consider another theory in the $\mathbb{H}_{\text{in bundle}}$ Hierarchy.

Proposition 22.

$T_{\text{gangemi}} \cup \Delta_{\text{gangemi2inbundle}} \models T_{\text{flag bundle}}$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/gangemi_participates/interprets/gangemi2flag_bundle/output/

Proposition 23.

$T_{\text{flag bundle}} \cup \Pi_{\text{inbundle2gangemi}} \models T_{\text{gangemi}}$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/gangemi_participates/interprets/flag_bundle2gangemi/output/

Theorem 24. $T_{\text{gangemi}}$ is synonymous with $T_{\text{flag bundle}}$. 
Proof. By Proposition 22, $T_{gangemi}$ faithfully interprets $T_{flag\_bundle}$; this interpretation is faithful because $T_{gangemi} \cup \Delta_{gangemi2inbundle}$ is a conservative extension of $T_{flag\_bundle}$.

By Proposition 23, $T_{flag\_bundle}$ faithfully interprets $T_{gangemi}$; this interpretation is faithful because $T_{flag\_bundle} \cup \Pi_{inbundle2gangemi}$ is a conservative extension of $T_{gangemi}$. □

Corollary 11. $\text{Mod}(T_{gangemi})$ is representable by $\mathcal{M}_{flag\_bundle}$.

Thus there is a one-to-one correspondence between the models of $T_{gangemi}$ and the models of $T_{flag\_bundle}$.

3.3. Beyond Incidence Bundles. We turn now to the verification of the third participation ontology, namely $T_{dolce\_participation}$ (see Figure 8). Following the approach in the preceding sections, it appears that the theory $T_{plane\_flag\_bundle}$ is the right theory to use.

Definition 27. $\Delta_{dolce2inbundle}$ is the set of translation definitions

$(\forall x)\ point(x) \equiv T(x)$
$(\forall x)\ line(x) \equiv ED(x)$
$(\forall x)\ plane(x) \equiv PD(x)$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
$(\forall x, y)in(x, y) \equiv$
We can use this Theorem to partially characterize models of $T_{\text{dolce-participation}}$ with respect to incidence bundles. Nevertheless, there is something more going on here, because $T_{\text{plane-flag bundle}}$ does not interpret $T_{\text{dolce-participation}}$. In particular, participation is constrained by the mereology on time within DOLCE. We will therefore need to introduce a new class of mathematical structures that extends incidence bundles with a mereology on the set of points.

4. Incidence Foliations

We now extend the first motivating scenario by adding the temporal aspect of processes to illustrate how a new set of mathematical structures, known as incidence foliations, can aid us in the verification of the notions of participation in DOLCE.

4.1. Motivating Scenario II. Suppose we complicate the baking example with a more complex recipe that involves multiple processes that depend on time (see Figure 6). The pastry chef wishes to bake a cake; for this example, assume the dry and wet ingredients have already been mixed (similar to the process presented in Section 2.1). In order to make a cake, the chef is required to put the cake batter ($x_1$) into a pan ($o_1$), and then bake the cake ($x_2$) for a required amount of time ($o_2$); during the baking process ($o_3$), the chef would like to make the frosting using a set of ingredients ($x_3$). The baking and frosting preparation processes overlap as the chef would like to minimize the amount of time it takes to prepare the cake. Because we need to capture the relationships of containment and overlap of the time intervals, there is a need for a mereology on the time intervals.

4.2. Introducing Foliations. An incidence foliation is an amalgamation of a mereological geometry and an incidence bundle (which is specified on each set of coincident lines and planes). Mereological geometries are analogous to ordered geometries [8]. In an ordered geometry, there is a bipartite incidence structure and betweenness relation on points; in addition, sets of collinear points satisfy stronger conditions with respect to the betweenness relation. In a mereological geometry, there is also a bipartite incidence structure, but instead of a betweenness relation, there is a mereology on the set of points, and collinear points satisfy stronger conditions within the mereology.

The hierarchy of theories that axiomatize different classes of incidence foliations can be found in Figure 7. In the remainder of this section, we will explore the theories within this hierarchy. Each theory in $\mathbb{H}_{\text{in-foliation}}$ is a conservative extension of an incidence bundle theory and a mereological geometry theory; in addition, it is a nonconservative extension of the root theory of incidence foliations with axioms that constrain the relationship between the incidence bundle and the mereology on points. We will see that this relationship among the subtheories is reflected in the relationship between the substructures of the models of the theory.

4.3. Root Theory $T_{\text{in-foliation-root}}$.

4.3.1. Axiomatization of $T_{\text{in-foliation-root}}$.

Definition 28.

$$T_{\text{in-foliation-root}} = T_{\text{wmg}} \cup T_{\text{in-bundle}}$$

Proposition 25. $T_{\text{in-foliation-root}}$ is consistent.
Proof. A model constructed by Mace4 can be found at colore.oor.net/incidence_foliation/consistency/output/in_foliation_root.

Proposition 26.

\[ T_{\text{in foliation root}} \models T_{\text{partial in bundle}} \]

Proof. http://... □

4.3.2. Characterization of \( \mathcal{M}_{\text{in foliation root}} \).

Definition 29. \( \mathcal{M} \in \mathcal{M}_{\text{in foliation root}} \) iff

\[ \mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{part}, \text{tin} \rangle \] such that

(1) \( \mathcal{M} = \langle P, L, \text{point}, \text{line}, \text{in}, \text{part} \rangle \in \mathcal{M}_{\text{wmg}} \);

(2) \( \mathcal{N} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathcal{M}_{\text{partial in bundle}} \).

4.3.3. Representation Theorem for \( \text{Mod}(T_{\text{in foliation root}}) \).

Theorem 26. \( \mathcal{M} \in \text{Mod}(T_{\text{in foliation root}}) \) iff \( \mathcal{M} \in \mathcal{M}_{\text{in foliation root}} \).

Proof. □
Theorem 27. \( T_{\text{in\_foliation\_root}} \) is reducible to \( T_{\text{wmg}} \) and \( T_{\text{partial\_in\_bundle}} \).

Proof.

4.4. \( T_{\text{downward\_in\_foliation}} \). Within models of \( T_{\text{downward\_in\_foliation}} \), there is a close relationship between the fibres of the incidence bundle and the mereology on the set of points. Recall that each fibre is associated with a set of pairs of planes and lines (which intuitively are temporarily incident); the condition in models of \( T_{\text{downward\_in\_foliation}} \) requires that the pairs associated with one point are also associated with all points which are its parts.
4.4.1. Axiomatization of $T_{\text{downward in foliation}}$.

**Definition 30.** $T_{\text{downward in foliation}}$ is the set of axioms in $T_{\text{in foliation root}}$ extended with the sentence

\[(\forall x, y, t_1, t_2) \ tin(x, y, t_1) \land \part(t_2, t_1) \supset tin(x, y, t_2)\]

**Proposition 27.** $T_{\text{downward in foliation}}$ is consistent.

*Proof.* A model constructed by Mace4 can be found at colore.oom.net/incidence_foliation/consistency/output/downward_in_foliation.

4.4.2. Characterization of $M_{\text{downward in foliation}}$.

**Definition 31.** $M \in M_{\text{downward in foliation}}$ iff

1. $M \in M_{\text{in foliation root}}$;
2. If $\langle x, y \rangle \in \part$, then $\pi^{-1}(y) \subseteq \pi^{-1}(x)$.

**Definition 32.** A set of points is coincident iff all points in the set are both collinear and coplanar.

**Definition 33.** Let $I = (Q, L, P, \in)$ be a tripartite incidence structure. The coincident graph $G = (P, E)$ for $I$ is the graph such that $(x, y) \in E$ iff $x$ and $y$ are coincident in $I$.

**Theorem 28.** Suppose $M \in M_{\text{in foliation}}$.

$M \in M_{\text{downward in foliation}}$ iff a subset of the set of the set of cliques in the coincident graph for the incidence structure $I$ in $M$ is a lower set in $\langle P, \part \rangle$.

*Proof.*

**Definition 34.** Let $P = (P, \part)$ be a mereology. The lower set graph $G = (P, E)$ for $P$ is the graph such that $(x, y) \in E$ iff $x$ and $y$ are elements of the same lower set in $P$.

4.4.3. Representation Theorems for Mod($T_{\text{downward in foliation}}$).

**Theorem 29.** $M \in \text{Mod}(T_{\text{downward in foliation}})$ iff $M \in M_{\text{downward in foliation}}$.

*Proof.* $\Rightarrow$)

Suppose $M \in \text{Mod}(T_{\text{downward in foliation}})$. Since

$T_{\text{downward in foliation}} \models T_{\text{in foliation}}$

we have $M \in \text{Mod}(T_{\text{in foliation}})$ by Theorem 27. Thus, condition (1) in Definition 31 is satisfied.

$M \models (\forall x, y, t_1, t_2) \ tin(x, y, t_1) \land \part(t_2, t_1) \supset tin(x, y, t_2)$

iff for any $t_1, t_2$ such that $(t_2, t_1) \in \part$ we have

$\langle x, y, t_1 \rangle \in \tin \Rightarrow \langle x, y, t_2 \rangle \in \tin$

By Definition of incidence bundles, this is equivalent to

$(x, y) \in \pi^{-1}(t_1) \Rightarrow (x, y) \in \pi^{-1}(t_2)$
that is, \( \pi^{-1}(t_1) \subseteq \pi^{-1}(t_2) \). Thus, condition (2) in Definition 31 is satisfied and \( M \in \mathfrak{M}^{\downarrow \text{infoliation}} \).

\[ \Leftarrow \]

Suppose \( M \in \mathfrak{M}^{\downarrow \text{infoliation}} \).

By condition (1) in Definition 31 and Theorem 26:

\[ T_{\downarrow \text{infoliation}} \models T_{\text{infoliation}} \]

Thus, condition (2) in Definition 31:

\[ (x, y) \in \pi^{-1}(t_1) \Rightarrow (x, y) \in \pi^{-1}(t_2) \]

for any \( t_1, t_2 \) such that \( (t_2, t_1) \in \text{part} \).

By Definition of incidence bundles, this is equivalent to

\[ (x, y, t_1) \in \text{tin} \Rightarrow (x, y, t_2) \in \text{tin} \]

so that

\[ M \models (\forall x, y, t_1, t_2) \text{tin}(x, y, t_1) \land \text{part}(t_2, t_1) \supset \text{tin}(x, y, t_2) \]

and hence \( M \in \text{Mod}(T_{\downarrow \text{infoliation}}) \).\( \square \)

4.5. \( T_{\uparrow \text{infoliation}} \).

**Definition 35.** \( T_{\uparrow \text{infoliation}} \) is the set of axioms in \( T_{\text{infoliation}} \) with the sentence

\[ (\forall x, y, t_1, t_2) \text{tin}(x, y, t_1) \land \text{part}(t_2, t_1) \supset \text{tin}(x, y, t_2) \]

**Proposition 28.** \( T_{\uparrow \text{infoliation}} \) is consistent.

**Proof.** A model constructed by Mace4 can be found at

\[ \text{colo.re.oor.net/incidence_foliation/consistency/output/upward_in_foliation} \text{model} \]

4.5.1. Characterization of \( \mathfrak{M}^{\uparrow \text{infoliation}} \).

**Definition 36.** \( M \in \mathfrak{M}^{\uparrow \text{infoliation}} \) iff

\[ M = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{part}, \text{tin} \rangle \text{ such that} \]

(1) \( M \in \mathfrak{M}^{\text{infoliation}_\text{root}} \).

(2) If \( (x, y) \in \text{part} \), then \( \pi^{-1}(x) \subseteq \pi^{-1}(y) \).

**Theorem 30.** Suppose \( M \in \mathfrak{M}^{\text{infoliation}} \).

\( M \in \mathfrak{M}^{\uparrow \text{infoliation}} \) iff a subset of the set of the set of cliques in the co-

incident graph for the incidence structure \( I \) in \( M \) is an upper set in \( \langle P, \text{part} \rangle \).

**Proof.** \( \Rightarrow \)

Suppose \( M \in \text{Mod}(T_{\uparrow \text{infoliation}}) \).

Since

\[ T_{\uparrow \text{infoliation}} \models T_{\text{infoliation}} \]

we have \( M \in \text{Mod}(T_{\text{infoliation}}) \) by Theorem 26. Thus, condition (1) in Definition 36 is satisfied.\( \square \)

4.5.2. Representation Theorems for \( \text{Mod}(T_{\uparrow \text{infoliation}}) \).

**Theorem 31.** \( M \in \text{Mod}(T_{\uparrow \text{infoliation}}) \), iff \( M \in \mathfrak{M}^{\uparrow \text{infoliation}} \).

**Proof.** \( \Rightarrow \)

Suppose \( M \in \text{Mod}(T_{\uparrow \text{infoliation}}) \).

Since

\[ T_{\uparrow \text{infoliation}} \models T_{\text{infoliation}} \]

we have \( M \in \text{Mod}(T_{\text{infoliation}}) \) by Theorem 26. Thus, condition (1) in Definition 36 is satisfied.\( \square \)
\[
\mathcal{M} \models (\forall x, y, t_1, t_2) \ tin(x, y, t_1) \land part(t_1, t_2) \supset tin(x, y, t_2)
\]
iff for any \( t_1, t_2 \) such that \( (t_1, t_2) \in \text{part} \) we have
\[
\langle x, y, t_1 \rangle \in \text{tin} \Rightarrow \langle x, y, t_2 \rangle \in \text{tin}
\]
By Definition of incidence bundles, this is equivalent to
\[
(x, y) \in \pi^{-1}(t_1) \Rightarrow (x, y) \in \pi^{-1}(t_2)
\]
that is, \( \pi^{-1}(t_1) \subseteq \pi^{-1}(t_2) \). Thus, condition (2) in Definition 36 is satisfied and \( \mathcal{M} \in \mathfrak{M}_{\text{upward in foliation}} \).
\[
\Rightarrow:
\]
Suppose \( \mathcal{M} \in \mathfrak{M}_{\text{upward in foliation}} \).
By condition (1) in Definition 36 and Theorem 26,
\[
\mathcal{T}_{\text{upward in foliation}} \models \mathcal{T}_{\text{in foliation}}
\]
Thus, condition (2) in Definition 36,
\[
(x, y) \in \pi^{-1}(t_1) \Rightarrow (x, y) \in \pi^{-1}(t_2)
\]
for any \( t_1, t_2 \) such that \( (t_1, t_2) \in \text{part} \).
By Definition of incidence bundles, this is equivalent to
\[
\langle x, y, t_1 \rangle \in \text{tin} \Rightarrow \langle x, y, t_2 \rangle \in \text{tin}
\]
so that
\[
\mathcal{M} \models (\forall x, y, t_1, t_2) \ tin(x, y, t_1) \land part(t_1, t_2) \supset tin(x, y, t_2)
\]
and hence \( \mathcal{M} \in \text{Mod}(\mathcal{T}_{\text{upward in foliation}}) \).

4.6. \( \mathcal{T}_{\text{plane downward in foliation}} \)

4.6.1. Axiomatization of \( \mathcal{T}_{\text{plane downward in foliation}} \)

Definition 37.
\[
\mathcal{T}_{\text{plane downward in foliation}} = \mathcal{T}_{\text{downward in foliation}} \cup \mathcal{T}_{\text{plane flag bundle}}
\]

Proposition 29. \( \mathcal{T}_{\text{plane downward in foliation}} \) is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/incidence_foliation/consistency/output/plane_downward_in_foliation.model

Proposition 30.
\[
\mathcal{T}_{\text{plane downward in foliation}} \models \mathcal{T}_{\text{partial plane flag bundle}}
\]

Proof. http://...

4.6.2. Characterization of \( \mathfrak{M}_{\text{plane downward in foliation}} \).

Definition 38. \( \mathcal{M} \in \mathfrak{M}_{\text{plane downward in foliation}} \) iff
\[
\mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{part}, \text{tin} \rangle \text{ such that }
\]
(1) \( \mathcal{M} \in \mathfrak{M}_{\text{downward in foliation}} \),
(2) \( \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathfrak{M}_{\text{partial plane flag bundle}} \).
4.6.3. Representation Theorems for Mod($T_{\text{plane\-downward\-in\-foliation}}$).

**Theorem 32.** $\mathcal{M} \in \text{Mod}(T_{\text{plane\-downward\-in\-foliation}})$ iff $\mathcal{M} \in \mathfrak{M}_{\text{plane\-downward\-in\-foliation}}$.

**Proof.** $\Rightarrow$:
Suppose $\mathcal{M} \in \text{Mod}(T_{\text{plane\-downward\-in\-foliation}})$.

Since

$$T_{\text{plane\-downward\-in\-foliation}} \models T_{\text{downward\-in\-foliation}}$$

we have $\mathcal{M} \in \mathfrak{M}_{\text{downward\-in\-foliation}}$ by Theorem 29 and condition (1) in Definition 38 is satisfied.

By Proposition 30 and Theorem 18 we have $\mathcal{M} \in \mathfrak{M}_{\text{partial\-plane\-flag\-bundle}}$ and condition (2) in Definition 38 is satisfied.

Therefore $\mathcal{M} \in \mathfrak{M}_{\text{plane\-downward\-in\-foliation}}$.

$\Leftarrow$:
Suppose $\mathcal{M} \in \mathfrak{M}_{\text{plane\-downward\-in\-foliation}}$.

By condition (1) in Definition 38 and Theorem 29

$$\mathcal{M} \models T_{\text{downward\-in\-foliation}}$$

By condition (2) in Definition 38 and Theorem 18

$$\mathcal{M} \models T_{\text{plane\-flag\-bundle}}$$

By definition,

$$\mathcal{M} \models T_{\text{plane\-downward\-in\-foliation}}$$

□

4.7. $T_{\text{line\-downward\-in\-foliation}}$.

4.7.1. Axiomatization of $T_{\text{line\-downward\-in\-foliation}}$.

**Definition 39.**

$$T_{\text{line\-downward\-in\-foliation}} = T_{\text{downward\-in\-foliation}} \cup T_{\text{line\-flag\-bundle}}$$

**Proposition 31.** $T_{\text{line\-downward\-in\-foliation}}$ is consistent.

**Proof.** A model constructed by Mace4 can be found at colore.oor.net/incidence_foliation/consistency/output/line_downward_in_foliation.model □

**Proposition 32.**

$$T_{\text{line\-downward\-in\-foliation}} \models T_{\text{partial\-line\-flag\-bundle}}$$

**Proof.** http://... □

4.7.2. Characterization of $\mathfrak{M}_{\text{line\-downward\-in\-foliation}}$.

**Definition 40.** $\mathcal{M} \in \mathfrak{M}_{\text{line\-downward\-in\-foliation}}$ iff

$\mathcal{M} = \langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{part}, \text{tin} \rangle$ such that

1. $\mathcal{M} \in \mathfrak{M}_{\text{downward\-in\-foliation}}$,
2. $\langle P, L, Q, \text{point}, \text{line}, \text{plane}, \text{in}, \text{tin} \rangle \in \mathfrak{M}_{\text{line\-flag\-bundle}}$. 

4.7.3. Representation Theorems for $\text{Mod}(T_{\text{line}\downarrow\text{in foliation}})$.

**Theorem 33.** $\mathcal{M} \in \text{Mod}(T_{\text{line}\downarrow\text{in foliation}})$ iff $\mathcal{M} \in \mathfrak{M}_{\text{line}\downarrow\text{in foliation}}$.

**Proof.** $\Rightarrow$: Suppose $\mathcal{M} \in \text{Mod}(T_{\text{line}\downarrow\text{in foliation}})$. Since

$$T_{\text{line}\downarrow\text{in foliation}} \models T_{\text{downward in foliation}}$$

we have $\mathcal{M} \in \mathfrak{M}_{\text{downward in foliation}}$ by Theorem 29 and condition (1) in Definition 40 is satisfied.

By Proposition 32 and Theorem 20, we have $\mathcal{M} \in \mathfrak{M}_{\text{partial line flag bundle}}$ and condition (2) in Definition 40 is satisfied. Therefore $\mathcal{M} \in \mathfrak{M}_{\text{line}\downarrow\text{in foliation}}$.

$\Leftarrow$: Suppose $\mathcal{M} \in \mathfrak{M}_{\text{line}\downarrow\text{in foliation}}$.

By condition (1) in Definition 40 and Theorem 29,

$$\mathcal{M} \models T_{\text{downward in foliation}}$$

By condition (2) in Definition 40 and Theorem 20,

$$\mathcal{M} \models T_{\text{line flag bundle}}$$

By definition,

$$\mathcal{M} \models T_{\text{line}\downarrow\text{in foliation}}$$

$\square$

4.8. $T_{\text{ideal cem}\downarrow\text{in foliation}}$.

4.8.1. Axiomatization of $T_{\text{ideal cem}\downarrow\text{in foliation}}$.

**Definition 41.**

$$T_{\text{ideal cem}\downarrow\text{in foliation}} = T_{\text{downward in foliation}} \cup T_{\text{ideal cem wmg}}$$

**Proposition 33.** $T_{\text{ideal cem}\downarrow\text{in foliation}}$ is consistent.

**Proof.** A model constructed by Mace4 can be found at

[https://colore.oor.net/incidence_foliation/consistency/output/ideal_cem_downward_in_foliation.model](https://colore.oor.net/incidence_foliation/consistency/output/ideal_cem_downward_in_foliation.model) $\square$

4.8.2. Characterization of $\mathfrak{M}_{\text{ideal cem}\downarrow\text{in foliation}}$.

**Definition 42.** $\mathcal{M} \in \mathfrak{M}_{\text{ideal cem}\downarrow\text{in foliation}}$ iff

$$\mathcal{M} = \langle P, L, \text{point, line, plane, in, part, tin} \rangle$$

such that

- $\mathcal{M} \in \mathfrak{M}_{\text{downward in foliation}}$;
- $\langle P, L, \text{point, line, in, part} \rangle \in \mathfrak{M}_{\text{ideal cem wmg}}$. 
4.8.3. Representation Theorems for \( \text{Mod}(T_{\text{ideal cem downward in foliation}}) \).

**Theorem 34.** \( \mathcal{M} \in \text{Mod}(T_{\text{ideal cem downward in foliation}}) \) iff \( \mathcal{M} \in \mathcal{M}_{\text{ideal cem downward in foliation}} \).

**Proof.** \( \Rightarrow \): Suppose \( \mathcal{M} \in \text{Mod}(T_{\text{ideal cem downward in foliation}}) \).

Since

\[ T_{\text{ideal cem downward in foliation}} \models T_{\text{downward in foliation}} \]

we have \( \mathcal{M} \models \mathcal{M}_{\text{downward in foliation}} \) by Theorem 29 and condition (1) in Definition 42 is satisfied.

Since

\[ T_{\text{ideal cem downward in foliation}} \models T_{\text{cem wmg}} \]

we have \( \mathcal{M} \models \mathcal{M}_{\text{ideal cem wmg}} \) and condition (2) in Definition 42 is satisfied.

Therefore \( \mathcal{M} \models \mathcal{M}_{\text{ideal cem downward in foliation}} \).

\( \Leftarrow \): Suppose \( \mathcal{M} \models \mathcal{M}_{\text{ideal cem downward in foliation}} \).

By condition (1) in Definition 42 and Theorem 29,

\[ \mathcal{M} \models T_{\text{downward in foliation}} \]

By condition (2) in Definition 42,

\[ \mathcal{M} \models T_{\text{ideal cem wmg}} \]

By definition,

\[ \mathcal{M} \models T_{\text{ideal cem downward in foliation}} \]

\[ \square \]

4.9. \( T_{\text{ideal cem plane downward in foliation}} \).

4.9.1. Axiomatization of \( T_{\text{ideal cem plane downward in foliation}} \).

**Definition 43.**

\[ T_{\text{ideal cem plane downward in foliation}} = T_{\text{plane downward in foliation}} \cup T_{\text{ideal cem wmg}} \]

**Proposition 34.** \( T_{\text{ideal cem plane downward in foliation}} \) is consistent.

**Proof.** A model constructed by Mace4 can be found at

[link]

\[ \square \]

4.9.2. Characterization of \( \mathcal{M}_{\text{ideal cem plane downward in foliation}} \).

**Definition 44.** \( \mathcal{M} \in \mathcal{M}_{\text{ideal cem plane downward in foliation}} \) iff \( \mathcal{M} = \langle P, L, \text{point, line, plane, in, part, tin} \rangle \) such that

- \( \mathcal{M} \in \mathcal{M}_{\text{plane downward in foliation}} \),
- \( \langle P, L, \text{point, line, in, part} \rangle \in \mathcal{M}_{\text{ideal cem wmg}} \).
4.9.3. Representation Theorems for Mod($T\text{ideal}_c\text{em}_{\text{plane downward in foliation}}$).

**Theorem 35.** $\mathcal{M} \in \text{Mod}(T\text{ideal}_c\text{em}_{\text{plane downward in foliation}})$ iff $\mathcal{M} \in \mathfrak{M}_{\text{ideal}_c\text{em}_{\text{plane downward in foliation}}}$.

**Proof.** $\Rightarrow$)
Suppose $\mathcal{M} \in \text{Mod}(T\text{ideal}_c\text{em}_{\text{plane downward in foliation}})$. Since

\[ T_{\text{ideal}_c\text{em}_{\text{plane downward in foliation}}} \models T_{\text{downward in foliation}} \]

we have $\mathcal{M} \in \mathfrak{M}_{\text{plane downward in foliation}}$ by Theorem 29 and condition (1) in Definition 42 is satisfied. Since

\[ T_{\text{ideal}_c\text{em}_{\text{plane downward in foliation}}} \models T_{\text{ideal}_c\text{em}_{\text{wmg}}} \]

we have $\mathcal{M} \in \mathfrak{M}_{\text{ideal}_c\text{em}_{\text{wmg}}}$ and condition (2) in Definition 42 is satisfied. Therefore $\mathcal{M} \in \mathfrak{M}_{\text{ideal}_c\text{em}_{\text{plane downward in foliation}}}$.

$\Leftarrow$)
Suppose $\mathcal{M} \in \mathfrak{M}_{\text{ideal}_c\text{em}_{\text{plane downward in foliation}}}$. By condition (1) in Definition 42 and Theorem 29

\[ \mathcal{M} \models T_{\text{downward in foliation}} \]

By condition (2) in Definition 42

\[ \mathcal{M} \models T_{\text{ideal}_c\text{em}_{\text{wmg}}} \]

By definition,

\[ \mathcal{M} \models T_{\text{ideal}_c\text{em}_{\text{plane downward in foliation}}} \]

\[ \square \]

4.10. $T_{\text{ideal}_c\text{em}_{\text{line downward in foliation}}}$.

4.10.1. Axiomatization of $T_{\text{ideal}_c\text{em}_{\text{line downward in foliation}}}$.

**Definition 45.**

\[ T_{\text{ideal}_c\text{em}_{\text{line downward in foliation}}} = T_{\text{line downward in foliation}} \cup T_{\text{ideal}_c\text{em}_{\text{wmg}}} \]

**Proposition 35.** $T_{\text{ideal}_c\text{em}_{\text{line downward in foliation}}}$ is consistent.

**Proof.** A model constructed by Mace4 can be found at [colore.oor.net/incidence_foliation/consistency/output/ideal_cem_line_downward_in_foliation.model] \[ \square \]

4.10.2. Characterization of $\mathfrak{M}_{\text{ideal}_c\text{em}_{\text{line downward in foliation}}}$.

**Definition 46.** $\mathcal{M} \in \mathfrak{M}_{\text{ideal}_c\text{em}_{\text{line downward in foliation}}}$ iff $\mathcal{M} = \langle P, L, \text{point}, \text{line}, \text{plane}, \text{in}, \text{part}, \text{tin} \rangle$ such that

- $\mathcal{M} \in \mathfrak{M}_{\text{line downward in foliation}}$,
- $\langle P, L, \text{point}, \text{line}, \text{in} \rangle \in \mathfrak{M}_{\text{ideal}_c\text{em}_{\text{wmg}}}$.\[ \square \]
4.10.3. **Representation Theorems for** $\text{Mod}(T_{\text{ideal cem line downward in foliation}})$.

**Theorem 36.** $\mathcal{M} \in \text{Mod}(T_{\text{ideal cem line downward in foliation}})$ iff $\mathcal{M} \in \mathbb{M}_{\text{ideal cem line downward in foliation}}$.

**Proof.** $\Rightarrow$)

Suppose $\mathcal{M} \in \text{Mod}(T_{\text{ideal cem line downward in foliation}})$.

Since $T_{\text{ideal cem line downward in foliation}} \models T_{\text{line downward in foliation}}$ we have $\mathcal{M} \in \mathbb{M}_{\text{line downward in foliation}}$ by Theorem 29 and condition (1) in Definition 42 is satisfied.

Since $T_{\text{ideal cem line downward in foliation}} \models T_{\text{ideal cem wmg}}$ we have $\mathcal{M} \in \mathbb{M}_{\text{ideal cem wmg}}$ and condition (2) in Definition 42 is satisfied.

$\Leftarrow$)

Suppose $\mathcal{M} \in \mathbb{M}_{\text{ideal cem line downward in foliation}}$.

By condition (1) in Definition 42 and Theorem 29

$\mathcal{M} \models T_{\text{line downward in foliation}}$

By condition (2) in Definition 42

$\mathcal{M} \models T_{\text{ideal cem wmg}}$

By definition,

$\mathcal{M} \models T_{\text{ideal cem line downward in foliation}}$

$\Box$

5. **Verification of** $T_{\text{dolce participation}}$

Now that we have a set of ontologies for incidence foliations, we can return to the verification of $T_{\text{dolce participation}}$. Intuitively, the mereological geometry in an incidence foliation corresponds to the subtheory of $T_{\text{dolce participation}}$ that axiomatizes the $\text{PRE}(x,t)$ relation between perdurants or endurants (which are interpreted by lines) and time intervals (which are interpreted as points). This raises a challenge. On the one hand, we have the problem that in the incidence bundle, both endurants and perdurants can be interpreted by lines, yet one class must be interpreted by planes in the incidence foliation. On the other hand, there is no other distinction between these two classes. As a result, we need two separate incidence foliations for the verification.

In $T_{\text{ideal cem plane downward in foliation}}$, we interpret endurants as planes and perdurants as lines within the incidence bundle. In the mereological geometry, there is a classical extensional mereology on the set of points, while sets of collinear points form ideals within the mereology.

**Lemma 10.** Let $\Delta_1$ be the set of translation definitions

$$
(\forall x) \text{point}^1(x) \equiv T(x)
$$

$$
(\forall x) \text{line}^1(x) \equiv ED(x)
$$

$$
(\forall x) \text{plane}^1(x) \equiv PD(x)
$$

$$
(\forall x,y) \text{in}^1(x,y) \equiv
$$
Proof.

Lemma 11. Let $\Delta_2$ be the set of translation definitions

\begin{align*}
(\forall x) \text{point}^2(x) &\equiv T(x) \\
(\forall x) \text{line}^2(x) &\equiv PD(x) \\
(\forall x) \text{plane}^2(x) &\equiv ED(x) \\
(\forall x, y) \text{in}^2(x, y) &\equiv \\
&((\text{PRE}(x, y) \land T(y)) \land (ED(x) \lor PD(x) \lor Q(x))) \\
&\lor((\text{PRE}(y, x) \land T(x)) \land (ED(y) \lor PD(y) \lor Q(y))) \\
&\lor((x = y) \land (ED(y) \lor PD(y) \lor Q(y)))) \\
(\forall x, y) \text{part}^2(x, y) &\equiv P(x, y) \land T(x) \land T(y) \\
(\forall x, y, z) \text{tin}^2(x, y, z) &\equiv PC(x, y, z)
\end{align*}

$T_{\text{dolce.participation}} \cup \Delta_2 \models T_{\text{ideal.cem.line.downward.in.foliation}}$

Figure 8. $T_{\text{dolce.participation}}$

\begin{align*}
&((\text{PRE}(x, y) \land T(y)) \land (ED(x) \lor PD(x) \lor Q(x))) \\
&\lor((\text{PRE}(y, x) \land T(x)) \land (ED(y) \lor PD(y) \lor Q(y))) \\
&\lor((x = y) \land (ED(y) \lor PD(y) \lor Q(y)))) \\
(\forall x, y) \text{part}^1(x, y) &\equiv P(x, y) \land T(x) \land T(y) \\
(\forall x, y, z) \text{tin}^1(x, y, z) &\equiv PC(x, y, z)
\end{align*}

$T_{\text{dolce.participation}} \cup \Delta_1 \models T_{\text{ideal.cem.plane.downward.in.foliation}}$

Proof. $\square$

In $T_{\text{ideal.cem.line.downward.in.foliation}}$, we interpret endurants as lines and perdurants as planes within the incidence bundle; the mereological geometry is the same as in $T_{\text{ideal.cem.plane.downward.in.foliation}}$.

Lemma 11. Let $\Delta_2$ be the set of translation definitions

\begin{align*}
(\forall x) \text{point}^2(x) &\equiv T(x) \\
(\forall x) \text{line}^2(x) &\equiv PD(x) \\
(\forall x) \text{plane}^2(x) &\equiv ED(x) \\
(\forall x, y) \text{in}^2(x, y) &\equiv \\
&((\text{PRE}(x, y) \land T(y)) \land (ED(x) \lor PD(x) \lor Q(x))) \\
&\lor((\text{PRE}(y, x) \land T(x)) \land (ED(y) \lor PD(y) \lor Q(y))) \\
&\lor((x = y) \land (ED(y) \lor PD(y) \lor Q(y)))) \\
(\forall x, y) \text{part}^2(x, y) &\equiv P(x, y) \land T(x) \land T(y) \\
(\forall x, y, z) \text{tin}^2(x, y, z) &\equiv PC(x, y, z)
\end{align*}

$T_{\text{dolce.participation}} \cup \Delta_2 \models T_{\text{ideal.cem.line.downward.in.foliation}}$
Lemma 12. Let $\Pi$ be the set of translation definitions

$$(\forall x, y, t) \ PC(x, y, t) \equiv \text{plane}(x) \land \text{line}(y) \land \text{point}(t) \land \text{in}(t, x) \land \text{tin}(x, y) \land \text{in}(t, y)$$

$$(\forall x, y) \ \text{PRE}(x, y) \equiv \text{((in}(y, x) \land \text{point}(y) \land \text{line}(x) \lor \text{plane}(x)))$$

$$(\forall x) \ T(x) \equiv \text{point}(x)$$

$$(\forall x) \ ED(x) \equiv \text{line}(x) \land L_1(x)$$

$$(\forall x) \ PD(x) \equiv \text{plane}(x) \land L_2(x)$$

$$(\forall x) \ Q(x) \equiv \text{line}(x) \land L_3(x)$$

$$(\forall x, y) \ P(x, y) \equiv \text{part}(x, y)$$

$T_{\text{ideal}} \cup T_{\text{line}} \models T_{\text{dolce participation}}$

Proof. By Lemma 10, $T_{\text{dolce participation}}$ interprets $T_{\text{ideal}} \cup \Delta_1$ is a conservative extension of $T_{\text{ideal}}$. Moreover, this interpretation is faithful because $T_{\text{dolce participation}} \cup \Delta_1$ is a conservative extension of $T_{\text{ideal}} \cup \Delta_1$.

By Lemma 11, $T_{\text{dolce participation}}$ interprets $T_{\text{ideal}} \cup \Delta_2$ is a conservative extension of $T_{\text{ideal}} \cup \Delta_2$.

By Lemma 12, $T_{\text{ideal}} \cup T_{\text{line}} \models T_{\text{dolce participation}}$. This interpretation is faithful because $T_{\text{ideal}} \cup T_{\text{line}}$ is a conservative extension $T_{\text{dolce participation}}$.

Finally, we have $T_{\text{dolce participation}} \cup \Delta_1 \cup \Delta_2 \models \Pi$

$T_{\text{ideal}} \cup T_{\text{line}} \models \Delta_1 \cup \Delta_2$

By [9], $T_{\text{dolce participation}}$ is synonymous with $T_{\text{ideal}} \cup T_{\text{line}}$.

Theorem 37. $T_{\text{dolce participation}}$ is synonymous with $T_{\text{ideal}} \cup T_{\text{line}}$

Proof. By the amalgamation of $M_{\text{ideal}}$ and $M_{\text{line}}$.
In other words, there is a one-to-one correspondence between models of $T_{\text{dolce\_participation}}$ and structures which are the amalgamation of models of $T_{\text{ideal\_cem\_plane\_downward\_in\_foliation}}$ and $T_{\text{ideal\_cem\_line\_downward\_in\_foliation}}$.

6. Incidence Bundles as Ontology Patterns

We have seen how incidence bundles have enabled the verification of the three participation ontologies. In this sense, the ontologies in $\mathbb{H}^{\text{in\_bundle}}$ Hierarchy serve as ontology patterns for the concept of participation, patterns which are instantiated through the notion of logical synonymy. Furthermore, we can exploit the synonymy of the different theories to support ontology design (through the notion of ontology transfer) and integration (by specifying the relationships among the ontologies).

6.1. Ontology Transfer. We can use incidence bundles to illustrate the notion of ontology transfer, in which we specify new ontologies in one hierarchy by identifying synonymous ontologies in another hierarchy. We will focus on the application of this technique to the specification of extensions of $T_{\text{psl\_participates}}$. On the left side of Figure 9 we can see the theories in the $\mathbb{H}^{\text{in\_bundle}}$ Hierarchy which we have used for verification of the participation ontologies. Using the translation definitions from the proof of Theorem 10, we can specify new ontologies which are extensions of $T_{\text{psl\_participates}}$ and which are synonymous with incidence bundle theories. Of particular interest are those theories of incidence bundles which were used in the verification of $T_{\text{participates\_owl}}$ and $T_{\text{dolce\_participation}}$.

Note that we use the same translation definitions that we used for $T_{\text{psl\_participates}}$.

6.1.1. $T_{\text{weak\_object\_exist}}$. If an object exists at some timepoint, then there is an activity that is occurring at that timepoint and in which the object participates.

**Definition 47.** $T_{\text{weak\_object\_exist}}$ is the set of axioms in $T_{\text{psl\_participates}}$ together with

$$(\forall x,t)\text{ object}(x) \land \text{exists\_at}(x,t) \supset (\exists o)\text{ participates\_in}(x,o,t)$$

**Proposition 36.** $T_{\text{weak\_object\_exist}}$ is consistent.

**Proof.** A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/weak_object_exist.model

**Proposition 37.**

$T_{\text{weak\_object\_exist}} \models (\forall x,t)\text{ object}(x) \land \text{exists\_at}(x,t) \supset (\exists o)\text{ is\_occurring\_at}(o,t)$

**Proof.** The proof generated by Prover9 can be found at colore.oor.net/psl_participates/theorems/output/weak_object_exist_1.proof

**Proposition 38.**

$T_{\text{weak\_object\_exist}} \cup \Delta_{\text{psl2inbundle}} \models T_{\text{wpl\_bundle}}$

The new ontologies can be found at http://colore.oor.net/psl_participates/
Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/weak_object_exist2wpl_bundle/output/

Proposition 39.

\[ T_{\text{wpl\_bundle}} \cup \Pi_{\text{inbundle2psl}} \models T_{\text{weak\_object\_exist}} \]

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/wpl_bundle2weak_object/output/

Theorem 39. \( T_{\text{weak\_object\_exist}} \) is synonymous with \( T_{\text{wpl\_bundle}} \).

Proof. By Proposition 38 \( T_{\text{weak\_object\_exist}} \) interprets \( T_{\text{wpl\_bundle}} \).

This interpretation is faithful because \( T_{\text{weak\_object\_exist}} \cup \Delta_{\text{psl2inbundle}} \) is a conservative extension of \( T_{\text{wpl\_bundle}} \).

By Proposition 39 \( T_{\text{wpl\_bundle}} \) interprets \( T_{\text{weak\_object\_exist}} \).

This interpretation is faithful because \( T_{\text{wpl\_bundle}} \cup \Pi_{\text{inbundle2psl}} \) is a conservative extension of \( T_{\text{weak\_object\_exist}} \).

By Proposition 20 and 9, \( T_{\text{weak\_object\_exist}} \) is synonymous with \( T_{\text{wpl\_bundle}} \).

Corollary 13. \( \text{Mod}(T_{\text{weak\_object\_exist}}) \) is representable by \( \mathfrak{M}_{\text{wpl\_bundle}} \).

6.1.2. \( T_{\text{object\_exist}} \). Every object participates in some activity occurrence at some timepoint.

Definition 48. \( T_{\text{object\_exist}} \) is the set of axioms in \( T_{\text{psl\_participates}} \) together with

(34) \((\forall x) \text{object}(x) \supset (\exists o,t \text{participates\_in}(x,o,t))\)

Proposition 40. \( T_{\text{object\_exist}} \) is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/object_exist.model

Proposition 41.

\[ T_{\text{object\_exist}} \models (\forall x) \text{object}(x) \supset (\exists o,t \text{exists\_at}(x,y) \land \text{is\_occurring\_at}(o,t)) \]

Proof. The proof generated by Prover9 can be found at colore.oor.net/psl_participates/theorems/output/object_exist_1.proof

Proposition 42.

\[ T_{\text{object\_exist}} \cup \Delta_{\text{psl}} \models T_{\text{pp\_bundle}} \]

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/object_exist2pp_bundle/output/

Proposition 43.

\[ T_{\text{pp\_bundle}} \cup \Pi_{\text{psl}} \models T_{\text{object\_exist}} \]
Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/pp_bundle2object_exist/output/□

**Theorem 40.** $T_{object\_exist}$ is synonymous $T_{pp\_bundle}$.

**Proof.** By Proposition 32 $T_{object\_exist}$ interprets $T_{pp\_bundle}$.

This interpretation is faithful because $T_{object\_exist} \cup \Delta_{psl2inbundle}$ is a conservative extension of $T_{pp\_bundle}$.

By Proposition 43 $T_{pp\_bundle}$ interprets $T_{object\_exist}$.

This interpretation is faithful because $T_{wpl\_bundle} \cup \Pi_{inbundle2psl}$ is a conservative extension of $T_{object\_exist}$.

By Proposition 20 and 9, $T_{object\_exist}$ is synonymous with $T_{pp\_bundle}$. □

**Corollary 14.** $Mod(T_{object\_exist})$ is representable by $\mathfrak{M}_{pp\_bundle}$.

6.1.3. $T_{weak\_actocc\_exist}$. If an activity is occurring at some timepoint, then there is an object that exists at that timepoint which participates in the activity occurrence.

**Definition 49.** $T_{weak\_actocc\_exist}$ is the set of axioms in $T_{psl\_participates}$ together with

(35) $(\forall o, t) activity\_occurrence(o) \land is\_occurring\_at(o, t) \supset (\exists x) participates\_in(x, o, t)$

**Proposition 44.** $T_{weak\_actocc\_exist}$ is consistent.

**Proof.** A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/weak_actocc_exist.model

**Proposition 45.** $T_{weak\_actocc\_exist} \models (\forall o, t) activity\_occurrence(o) \land is\_occurring\_at(o, t) \supset (\exists x) exists\_at(x, t)$

**Proof.** The proof generated by Prover9 can be found at colore.oor.net/psl_participates/theorems/output/weak_actocc_exist_1.proof

**Proposition 46.**

$$T_{weak\_actocc\_exist} \cup \Delta_{psl} \models T_{nip\_bundle}$$

**Proof.** Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/weak_actocc_exist2nip_bundle/output/□

**Proposition 47.**

$$T_{nip\_bundle} \cup \Pi_{psl} \models T_{weak\_actocc\_exist}$$

**Proof.** Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/nip_bundle2weak_actocc_exist/output/□

**Theorem 41.** $T_{weak\_actocc\_exist}$ is synonymous with $T_{nip\_bundle}$. 
Proof. By Proposition 46, \( T_{\text{weak,actocc,exist}} \) interprets \( T_{\text{nlp,bundle}} \).

This interpretation is faithful because \( T_{\text{weak,actocc,exist}} \cup \Delta_{\text{p2lmbundle}} \) is a conservative extension of \( T_{\text{nlp,bundle}} \).

By Proposition 47, \( T_{\text{nlp,bundle}} \) interprets \( T_{\text{weak,actocc,exist}} \).

This interpretation is faithful because \( T_{\text{nlp,bundle}} \cup \Pi_{\text{inbundle2psl}} \) is a conservative extension of \( T_{\text{weak,actocc,exist}} \).

By Proposition 20 and 9, \( T_{\text{weak,actocc,exist}} \) is synonymous with \( T_{\text{nlp,bundle}} \). \( \square \)

Corollary 15. \( \text{Mod}(T_{\text{weak,actocc,exist}}) \) is representable by \( \mathfrak{M}_{\text{nlp,bundle}} \).

6.1.4. \( T_{\text{actocc,exist}} \). Every activity occurrence has an object which participates in it at some timepoint.

Definition 50. \( T_{\text{actocc,exist}} \) is the set of axioms in \( T_{\text{psl,participates}} \) together with

\[
(\forall o) \text{activity occurrence}(o) \supset (\exists o,t) \text{participates in}(x,o,t)
\]

Proposition 48. \( T_{\text{actocc,exist}} \) is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/actocc_exist.model \( \square \)

Proposition 49.

\( T_{\text{actocc,exist}} \models (\forall o)\text{activity occurrence}(o) \supset (\exists x,t)\text{is occurring at}(o,t) \land \text{exists at}(x,t) \)

Proof. The proof generated by Prover9 can be found at colore.oor.net/psl_participates/theorems/output/actocc_exist_1.proof \( \square \)

Proposition 50.

\( T_{\text{actocc,exist}} \cup \Delta_{\text{psl}} \models T_{\text{lmbundle}} \)

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/actocc_exist2lmbundle/output/ \( \square \)

Proposition 51.

\( T_{\text{lmbundle}} \cup \Pi_{\text{psl}} \models T_{\text{actocc,exist}} \)

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/lmbundle2actocc_exist/output/ \( \square \)

Theorem 42. \( T_{\text{actocc,exist}} \) is synonymous \( T_{\text{lmbundle}} \).

Proof. By Proposition 50, \( T_{\text{actocc,exist}} \) interprets \( T_{\text{lmbundle}} \).

This interpretation is faithful because \( T_{\text{actocc,exist}} \cup \Delta_{\text{p2lmbundle}} \) is a conservative extension of \( T_{\text{lmbundle}} \).

By Proposition 51, \( T_{\text{lmbundle}} \) interprets \( T_{\text{actocc,exist}} \).

This interpretation is faithful because \( T_{\text{lmbundle}} \cup \Pi_{\text{inbundle2psl}} \) is a conservative extension of \( T_{\text{actocc,exist}} \).

By Proposition 20 and 9, \( T_{\text{actocc,exist}} \) is synonymous with \( T_{\text{lmbundle}} \). \( \square \)

Corollary 16. \( \text{Mod}(T_{\text{actocc,exist}}) \) is representable by \( \mathfrak{M}_{\text{lmbundle}} \).
6.1.5. $T_{\text{strong\_object\_exist}}$.

Definition 51.

$$T_{\text{strong\_object\_exist}} = T_{\text{object\_exist}} \cup T_{\text{weak\_actocc\_exist}}$$

Proposition 52. $T_{\text{strong\_object\_exist}}$ is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/strong_object_exist.

Proposition 53.

$$T_{\text{strong\_object\_exist}} \cup \Delta_{psl} \models T_{\text{plane\_flag\_bundle}}$$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/strong_object_exist2plane_flag_bundle/output/□

Proposition 54.

$$T_{\text{plane\_flag\_bundle}} \cup \Pi_{psl} \models T_{\text{strong\_object\_exist}}$$

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/plane_flag_bundle2strong_object_exist/output/□

Theorem 43. $T_{\text{strong\_object\_exist}}$ is synonymous with $T_{\text{pp\_bundle}}$.

Proof. By Proposition 53, $T_{\text{strong\_object\_exist}}$ interprets $T_{\text{plane\_flag\_bundle}}$.

This interpretation is faithful because $T_{\text{strong\_object\_exist}} \cup \Delta_{psl2inbundle}$ is a conservative extension of $T_{\text{plane\_flag\_bundle}}$.

By Proposition 54, $T_{\text{plane\_flag\_bundle}}$ interprets $T_{\text{strong\_object\_exist}}$.

This interpretation is faithful because $T_{\text{plane\_flag\_bundle}} \cup \Pi_{inbundle2psl}$ is a conservative extension of $T_{\text{strong\_object\_exist}}$.

By Proposition 20 and [9], $T_{\text{strong\_object\_exist}}$ is synonymous with $T_{\text{plane\_flag\_bundle}}$.

Corollary 17. Mod($T_{\text{strong\_object\_exist}}$) is representable by $T_{\text{plane\_flag\_bundle}}$.

6.1.6. $T_{\text{strong\_actocc\_exist}}$.

Definition 52.

$$T_{\text{strong\_actocc\_exist}} = T_{\text{weak\_object\_exist}} \cup T_{\text{actocc\_exist}}$$

Proposition 55. $T_{\text{strong\_actocc\_exist}}$ is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/strong_actocc_exist.

Proposition 56.

$$T_{\text{strong\_actocc\_exist}} \cup \Delta_{psl} \models T_{\text{line\_flag\_bundle}}$$
Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/strong_actocc_exist2line_flag_bundle/output/

Proposition 57.
\[ T_{\text{line flag bundle}} \cup \Pi_{\text{psl}} \models T_{\text{strong actocc exist}} \]

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/line_flag_bundle2strong_actocc_exist/output/

Theorem 44. \( T_{\text{strong actocc exist}} \) is synonymous with \( T_{\text{line flag bundle}} \).

Proof. By Proposition 56, \( T_{\text{strong actocc exist}} \) interprets \( T_{\text{line flag bundle}} \). This interpretation is faithful because \( T_{\text{strong actocc exist}} \cup \Delta_{\text{psl}} \subset \text{inbundle} \) is a conservative extension of \( T_{\text{line flag bundle}} \).

By Proposition 57, \( T_{\text{line flag bundle}} \) interprets \( T_{\text{strong actocc exist}} \). This interpretation is faithful because \( T_{\text{line flag bundle}} \cup \Pi_{\text{inbundle}2\text{psl}} \subset \text{psl} \) is a conservative extension of \( T_{\text{strong actocc exist}} \).

By Proposition 20 and [9], \( T_{\text{strong actocc exist}} \) is synonymous with \( T_{\text{line flag bundle}} \).

Corollary 18. \( \text{Mod}(T_{\text{strong actocc exist}}) \) is representable by \( M_{\text{line flag bundle}} \).

6.1.7. \( T_{\text{strong participates}} \).

Definition 53.
\[ T_{\text{strong participates}} = T_{\text{strong object exist}} \cup T_{\text{strong actocc exist}} \]

Proposition 58. \( T_{\text{strong participates}} \) is consistent.

Proof. A model constructed by Mace4 can be found at colore.oor.net/psl_participates/consistency/output/strong_participates/model/

Proposition 59.
\[ T_{\text{strong participates}} \cup \Delta_{\text{psl}} \models T_{\text{flag bundle}} \]

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/strong_participates2flag_bundle/output/

Proposition 60.
\[ T_{\text{flag bundle}} \cup \Pi_{\text{psl}} \models T_{\text{strong participates}} \]

Proof. Proofs generated by Prover9 can be found at colore.oor.net/psl_participates/interprets/flag_bundle2strong_participates/output/

Theorem 45. \( T_{\text{strong participates}} \) is synonymous with \( T_{\text{flag bundle}} \).
Figure 9. Relationship between extensions of $T_{psl\text{-participates}}$ and the hierarchy of incidence bundles. Dashed lines denote nonconservative extension, and solid lines denote logical synonymy.

**Proof.** By Proposition 59, $T_{strong\text{-participates}}$ interprets $T_{flag\text{-bundle}}$.

This interpretation is faithful because $T_{strong\text{-participates}} \cup \Delta_{psl2inbundle}$ is a conservative extension of $T_{flag\text{-bundle}}$.

By Proposition 60, $T_{flag\text{-bundle}}$ interprets $T_{strong\text{-participates}}$.

This interpretation is faithful because $T_{flag\text{-bundle}} \cup \Pi_{inbundle2psl}$ is a conservative extension of $T_{strong\text{-participates}}$.

By Proposition 20 and [9], $T_{strong\text{-participates}}$ is synonymous with $T_{flag\text{-bundle}}$. □

**Corollary 19.** $Mod(T_{strong\text{-participates}})$ is representable by $\mathfrak{M}_{flag\text{-bundle}}$.

6.1.8. **Summary.** Note that $T_{psl\text{-participates}}$ is the root theory of the $\mathbb{H}_{psl\text{-participates}}$ Hierarchy – there is no weaker theory than $T_{psl\text{-participates}}$ that axiomatizes intuitions about participation. Since $T_{present\text{-bundle}}$ is synonymous with $T_{psl\text{-participates}}$ (which is the root theory of its hierarchy), each theory in the $\mathbb{H}_{in\text{-bundle}}$ Hierarchy which extends it is synonymous with an extension of $T_{psl\text{-participates}}$ (see Figure 9).

6.2. **Relationships among Participation Ontologies.** We can use the different ontologies in the $\mathbb{H}_{in\text{-bundle}}$ Hierarchy to support the integration of the different participation ontologies. We have shown that $T_{dolce\text{-participates}}$ faithfully interprets $T_{plane\text{-flag\text{-bundle}}}$, and hence faithfully interprets $T_{strong\text{-object\text{-exist}}}$, which is a nonconservative extension of $T_{psl\text{-participates}}$. We have also shown $T_{participates\text{-owl}}$ is synonymous with $T_{flag\text{-bundle}}$, and hence faithfully interprets $T_{strong\text{-participates}}$, which is a nonconservative extension of $T_{strong\text{-object\text{-exist}}}$. In this way, we can compare the ontological commitments of each participation ontology by identifying their similarities within the $\mathbb{H}_{psl\text{-participates}}$ Hierarchy. In particular, we can
see that $T_{\text{gangemi}}$ is stronger than $T_{\text{dolce\_participates}}$, which in turn is stronger than $T_{\text{psl\_participates}}$.

7. Summary

We have introduced two new families of verified ontologies (incidence bundles in the $H_{\text{bundle}}$ Hierarchy and incidence foliations in the $H_{\text{foliation}}$ Hierarchy) which provide the mathematical foundations for three generic ontologies for participation. These mathematical ontologies play multiple roles within ontology design, reuse, and evaluation. First, they support the verification of the participation ontologies – not only can we show that the participation ontologies are consistent, but we can also characterize all models of the ontologies up to isomorphism. Second, since the verification is done in the context of an ontology repository, the ontologies for incidence bundles and foliations serve as ontology patterns which are instantiated by the participation ontologies through the notion of logical synonymy. Third, through the mappings used in the verification proofs, we can establish the relationships among the process ontologies and identify the similarities among their ontological commitments.

If we follow this approach a little further, we can see that all possible participation ontologies are implicitly specified as possible extensions of the root theory $T_{\text{in\_bundle}}$ of the $H_{\text{bundle}}$ Hierarchy (i.e. as different instantiations of the ontology pattern). These will either be nonconservative extensions of $T_{\text{in\_bundle}}$ if they share the same basic ontological commitments of $T_{\text{psl\_core\_participates}}$ and $T_{\text{participates\_owl}}$, or they will be conservative extensions that introduce new concepts, such as roles played by objects. For example, one can specify axioms that constrain how objects participate in different subactivity occurrences of an activity occurrence, or how different parts of an object can participate in an activity occurrence. Future work will pursue these directions for new participation ontologies.

References