# Model-Theoretic Characterization of Asher and Vieu's Ontology of Mereotopology

**Torsten Hahmann**
Department of Computer Science
University of Toronto
*torsten@cs.toronto.edu*

**Michael Gruninger**
Department of Mechanical and Industrial Engineering
Department of Computer Science
University of Toronto
*gruninger@mie.utoronto.ca*

**Abstract**

We characterize the models of Asher and Vieu's first-order mereotopology that evolved from Clarke's *Calculus of Individuals* in terms of mathematical structures with well-defined properties: topological spaces, lattices, and graphs. Although for the theory $RT$ soundness and completeness with respect to a topological translation of the axioms has been proved, this provides only sparse insights into the structural properties of the mereotopological models. We prove that the models of the subtheory $RT^-$ are isomorphic to p-ortholattices (pseudocomplemented, orthocomplemented). Combining the advantages of lattices and graphs, we give a representation theorem for the finite models of $RT^-_{EC}$ and show how to construct finite models of the full mereotopology. We compare our results to representations of the *Calculus of Individuals* and the *Region Connection Calculus* (RCC).

## 1 Introduction

Mereotopological systems have long been considered in philosophic and logic communities and recently received attention from a knowledge representation perspective. Mereotopology is composed of topological (from *point-set topology*) notions of connectedness and mereological notions of parthood. Point-set topology (or *General Topology*) relies on the definition of open (and dually closed) sets and extends standard set-theoretic notions of union, intersection, and containment with concepts such as interior, closure, limit points, neighborhoods, and connectedness.

Uncertainty about differences in mereotopological systems, in particular about their implicit [inherent] assumptions, seem to be a major source of confusion that hinders forthright application of even well-developed mereotopological theories. This problem arises in the various theories for different reasons: some lack any formal representation, leaving the user unsure about intended interpretations; others are formalized in first-order logic but lack a characterization of the models up to isomorphism. This paper focuses on an instance of the latter problem – we analyze the models of Asher and Vieu's mereotopology $RT_0$ (**?**) in the style of a representation theorem using structures from well-understood mathematical disciplines. We want to understand what kind of models the axiomatic system $RT_0$ describes and what properties these models share. The goal is to characterize the models of $RT_0$ in terms of classes of structures defined in topology, lattice theory, and graph theory and compare the classes to representations of other mereotopological theories. Although the completeness and soundness of $RT_0$ has been proven with respect to the intended models defined by $RT_T$ over a topology $T$, this is a mere rephrasing of the axioms. The proofs show that the axiomatic system describes exactly the intended models, but the formulation of the intended models does not reveal structural properties that can be used to learn about practical applicability, implicit restrictions, and hidden assumptions of the theory. [Special emphasis in our work is put on the finite models, since these are dominant in real-world applications.]

Primary motivation of this work is to give better insight into the axiomatic theory and to uncover problems and assumptions that users of the ontology should be aware of. A characterization of the models of the axiomatic theory allows us to reuse knowledge about the mathematical structures in the mereotopological theory. Afterwards, we can compare our results to the characterization of the *Region Connection Calculus* (RCC), as conducted in (**?**; **?**) which uses the notion of *Connection Algebras* to describe the *RCC*. Besides the characterization and analysis of $RT$, the main contribution of this work is a comparison of the suitability of different mathematical structures, in particular topological spaces, graph representations, and lattices, for a model-theoretic analysis and comparison of mereotopological frameworks. In the long-term an exhaustive comparison of different mereotopological approaches within a strictly defined mathematical context is desired. It turns out that lattices and lattices as graphs are best suited because lattices provide an intuitive way to model parthood relations.

Notice that we are only interested in a rigid mathematical study to provide the community with a model-theoretic view on mereotopology for the example of $RT_0$; we do not argue for or against underlying assumptions of different mereotopologies.

Serving the growing interesting in formal ontologies and upper ontologies, this kind of analysis can guide the selection of a generic axiomatization of mereotopology for inclusion in upper ontologies such as SUMO, DOLCE, and BFO.

## 2 The Mereotopology *RT* of Asher and Vieu

Mereology investigates parthood structures and relative complementation, dating back to Whitehead (?) and Leśniewski (?). The first formal specification of extensional mereology was (?). For an overview of extensional mereology, we invite the reader to consult (?). The primitive relation in mereology is *parthood* (an entity being part of another) expressed as irreflexive *proper parthood*, $<$ or *PP*, or as reflexive *parthood*, $\leq$ or *P*. The latter is usually a standard partial order that is reflexive, anti-symmetric, and transitive, coined *Ground Mereology* **M** in (?). Moreover, most mereologies define concepts of *overlap*, *union*, and *intersection* of entities. General sums (fusion), i.e. the union of arbitrarily many individuals, are also widespread. In all mereological theories a *whole* (universe) can be defined as the entity that everything else is part of. If differences are defined, a complement exists for every individual relative to the mereological whole. More controversial is whether mereology should allow *atoms*, i.e. individuals without proper parts that are the smallest entities of interest. Some theories are atomless while others explicitly force the existence of atoms (?); mereotopology inherits this controversy: it can be defined atomless, atomic, or make no assumption about atomism at all.

Neither topology nor mereology are by themselves powerful enough to express part-whole relations: Connection does not imply parthood between two individuals and disconnection does not prevent parthood as well as mereological wholes do not imply topological (self-connected) wholes. To be able to reason about integral, self-connected individuals, ways to combine mereology with topology are necessary. The different options to merge the two independent theories are used in (?) to classify mereotopologies. Bridging the gap between mereology and topology can be achieved by extending mereology with a topological primitive as applied in (?; ?; ?). More widespread is the reverse: assuming topology to be more fundamental and defining mereology on top of it using only topological primitives (“Topology as Basis for Mereology”). (?; ?; ?; ?) and the *RCC* (?; ?) use this method with a connection (or contact) relation as only primitive and parthood expressed in terms of connection. A third, less common way to merge topology and mereology was presented in (?) extending the mereological framework of (?) by quasi-mereological notions (combining mereology with some topological idea) to define topological wholeness.

As mentioned before, our focus are first-order mereotopologies. However, most of these theories either entirely lack soundness and completeness proofs, e.g. (?; ?; ?), or the proof is based on a rephrased model definition as in (?). Only the theory of (?), which is unfortunately limited to planar polygonal mereotopology, provides formal proofs that exhibit the possible models. For the *RCC* (?) the intended models are thoroughly described but the actual models not yet fully characterized. But to compare different mereotopologies by their models, we first need to characterize the models only from the axioms (or a definition for which equivalence to the axioms is proved). Clarke’s theory has received significant attention, but since some problems have been identified with it, we focus on Asher and Vieu’s version of the theory where the completeness and soundness proof ease the model-theoretic analysis. Notice that Clarke’s and Asher and Vieu’s theories are more sophisticated that the *RCC* which does not distinguish individuals from their interiors and closures, claiming such distinction superfluous for spatial reasoning. But contradictory, tangential and non-tangential parts as well as regular overlap and external connection which all rely on open and closed properties are distinguished in *RCC*.

### 2.1 Axiomatization $RT_0$

The first-order theory $RT_0$ of (?) uses the *connection* relation $C$ as only primitive. The theory is based on Clarke’s *Calculus of Individuals* (?; ?), with changes to make the theory first-order definable: (1) the explicit fusion operator is eliminated, it is claimed unnecessary; and (2) the concept of *weak contact, WCont*, is added. To eliminate trivial models, $RT_0$ requires at least one *external connection* and one *weak contact* (A11, A12). Some ontological and cognitive issues are also addressed, see (?). $RT_0$ follows the strategy called “Topology as Basis for Mereology” for defining mereotopology and does not contain an explicit mereology. Consequently, the parthood relation $P$ is defined solely in terms of the primitive $C$ which limits the whole theory to the expressiveness of $C$. For consequences of this kind of axiomatization, see (?; ?).

To construct models of the theory $RT_0$ the following definitions are necessary. Except for *WCont*, these were already defined in (?) and are comparable to those of other mereotopological systems.

(D1) $P(x,y) \equiv \forall z\,[C(z,x) \rightarrow C(z,y)]$ (Parthood as reflexive partial order satisfying the axioms of **M**)

(D3) $O(x,y) \equiv \exists z\,[P(z,x) \wedge P(z,y)]$ (Two individuals overlap iff they have a common part)

(D4) $EC(x,y) \equiv C(x,y) \wedge \neg O(x,y)$ (Two individuals are externally connected iff they are connected but share no common part)

(D6) $NTP(x,y) \equiv P(x,y) \wedge \neg\exists z\,[EC(z,x) \wedge EC(z,y)]$ (Non-tangential parts do not touch the border of the larger individuals)

(D8) $OP(x) \equiv x = i(x)$ (Open individuals)

(D9) $CL(x) \equiv x = c(x)$ (Closed individuals)

(D11) $WCont(x,y) \equiv \neg C(c(x),c(y)) \wedge \forall z\,[(OP(z) \wedge P(x,z)) \rightarrow C(c(z),y)]$ (Weak contact requires the closures of $x$ and $y$ to be disconnected, but any neighborhood containing $cl(x)$ to be connected to $y$)

The concepts proper part *PP* (the irreflexive subset of the extension of parthood, i.e. $PP(x,y) \equiv P(x,y) \wedge x \neq y$), tangential part *TP* ($TP(x,y) \equiv P(x,y) \wedge \neg NTP(x,y)$), and self-connectedness *CON* (see (?)) are defined in $RT_0$, but are irrelevant for the model construction, since they are not used in the axioms. $RT_0$ is then defined by:

(A1) $\forall x\,[C(x,x)]$ ($C$ reflexive)

(A2) $\forall x,y\,[C(x,y) \rightarrow C(y,x)]$ ($C$ symmetric)

(A3) $\forall x,y,z\,[(C(z,x) \equiv C(z,y)) \rightarrow x = y]$ ($C$ idempotent)

(A4) $\exists x \forall u\,[C(u,x)]$ (Universally connected element $a^* = x$)

(A5) $\forall x,y \exists z \forall u\, [C(u,z) \equiv (C(u,x) \vee C(u,y))]$ (Sum for pairs of elements)

(A6) $\forall x,y\, [O(x,y) \rightarrow \exists z \forall u\, [C(u,z) \equiv \exists v\, (P(v,x) \wedge P(v,y) \wedge C(v,u))]]$ (Intersection for pairs of overlapping element)

(A7) $\forall x\, [\exists y\, (\neg C(y,x)) \rightarrow \exists z \forall u\, [C(u,z) \equiv \exists v\, (\neg C(v,x) \wedge C(v,u))]]$ (Complement for elements $\neq a^*$)

(A8) $\forall x \exists y \forall u\, [C(u,y) \equiv \exists v\, (NTP(v,x) \wedge C(v,u))]$ (Interior for all elements; the interior $y = i(x)$ is the greatest non-tangential (not necessarily proper) part $y$ of $x$)

(A9) $c(a^*) = a^*$ (Closure $c$ defined as complete function)

(A10) $\forall x,y\, [(OP(x) \wedge OP(y) \wedge O(x,y)) \rightarrow OP(x \cap y)]$ (The intersection of open individuals is also open)

(A11) $\exists x,y\, [EC(x,y)]$ (Existence of two externally connected elements)

(A12) $\exists x,y\, [WCont(x,y)]$ (Existence of two elements in weak contact)

(A13) $\forall x \exists y\, [P(x,y) \wedge OP(y) \wedge \forall z\, [(P(x,z) \wedge OP(z)) \rightarrow P(y,z)]]$ (Unique open neighborhood for all elements)

We considered subtheories of the axioms of $RT_0$, which we refer to as $RT_C$, $RT^-$, and $RT^-_{EC}$. $RT_C$ is the topological core of the theory consisting of axioms A1 to A3. Extensional by A3, $RT_C$ corresponds to extensional ground topology (**T**) or Strong Mereotopology (**SMT**) (**?**) and to extensional weak contact algebras, satisfying axioms C0 - C3 and C5e of (**?**). Hence, $C$ is a contact relation in the sense of (**?**). $RT^- \equiv RT_0 \setminus \{A11, A12\}$ excludes the existential axioms that eliminate trivial models, but has the same structural properties as $RT_0$. A representation theorem for the models of $RT^-$ elegantly captures important properties of $RT_0$ as well. Finally, we consider models of $RT^-_{EC} \equiv RT^- \cup \{A11\}$ and show how external connections change the representation as lattices.

### 2.2 Intended Models $RT_T$

Asher and Vieu provide completeness and soundness proofs of $RT_0$ with respect to structures $RT_T$ that define the intended models of the mereotopology. Each intended model is build from a non-empty topological space $(X,T)$ with $T$ denoting the set of open sets of the space. Standard topological definitions of interior *int* and closure operators *cl*, open and closed properties, and $\sim$ as relative complement with respect to $X$ are assumed. The intended models are then defined as structures $RT_T = \langle Y, f, [\![ ]\!] \rangle$[1] where the set $Y$ must meet the conditions (i) to (viii). However, one can easily see that the conditions (i) to (viii) are a mere rephrasing of A4 to A13 of $RT_0$ and thus do not help to understand the theory in terms of well-known structures despite their common-sense motivation. Only the connections structures defined by $RT_C$ are not directly represented by the conditions (i) to (viii).

(i) $Y \subseteq \mathscr{P}(X)$ and $X \in Y$; $X$ is the universally connected individual $a^*$ required by A4 and all other elements in a model of $RT_0$ are subsets thereof;

*full interiors* (ii) and *smooth boundaries* (iii):

(ii) $\forall x \in Y\, (int(x) \in Y \,\&\, int(x) \neq \emptyset \,\&\, int(x) = int(cl(x)))$; requires non-empty interiors for all elements equivalent to A8;

[1]For definitions of $f$ and $[\![ ]\!]$, see (**?**)

(iii) $\forall x \in Y\, (cl(x) \in Y \,\&\, cl(x) = cl(int(x)))$; requires closures for all elements which is implicitly given by D7 as closure of the uniquely identified interiors and complements (by A7 and A8); A9 handles $a^*$ separately;

(iv) $\forall x \in Y\, (int(\sim x) \neq \emptyset \rightarrow \sim x \in Y)$; requires unique complements equivalent to A7;

(v) $\forall x,y \in Y\, (int(x \cap y) \neq \emptyset \rightarrow (x \cap^* y) \in Y)$; for pairs of elements with non-empty mereological intersection an intersecting element is guaranteed equivalent to A6;

(vi) $\forall x,y \in Y\, ((x \cup^* y) \in Y)$; guarantees the existence of sums of pairs equivalent to A5;

(vii) $\exists x,y \in Y\, ((x \cap y) \neq \emptyset \,\&\, int(x \cap y) = \emptyset)$; requires a pair of externally connected elements equivalent to A11 with def. D4;

(viii) $\exists x,y \in Y\, ((cl(x) \cap cl(y)) = \emptyset \;\&\, \forall z \in Y\, [(open(z) \,\&\, x \subseteq z) \rightarrow y \cap cl(z) \neq \emptyset])$; requires a pair of weakly connected elements equivalent to A12 with def. D11;

where $x \cup^* y = x \cup y \cup int(cl(x \cup y))$ and $x \cap^* y = x \cap y \cap cl(int(x \cap y))$.

This characterization of the intended models of $RT_0$ is insufficient for understanding properties and structure of the mereotopological models. The interplay of the conditions and resulting implicit constraints are not clear. Our goal is to better understand the models by characterizing them in the next section as classes of well-understood mathematical structures.

## 3 Characterization

This section presents our characterization of the models of $RT_0$ and subsets thereof in terms of topological spaces, lattices, graphs, and a combination of lattices and graphs. We are the first to characterize the models of a mereotopological or any spatial reasoning framework using all these different structures. Previously, (**?**) characterized the models of Clarke's *Calculus of Individuals* (**?**; **?**) in terms of lattices. showing that the *connection structures* defined by a subset of the axioms of Clarke (axioms A1 to A4) are isomorphic to the complete orthocomplemented lattices. Together with an axiom requiring the existence of a common point of two connected individuals, (**?**) proved that the *connection structures* are equivalent to the complete Boolean algebras. Since major problems have been observed with Clarke's *Calculus of Individuals*, a natural question is whether the system $RT_0$ of Asher and Vieu is an adequate replacement. $RT_0$ heavily relies on the work of Clarke; but it is not clear how the changes proposed by Asher and Vieu alter the class of associated models, particularly in a lattice-theoretic description. A by-product of our characterization is the extraction of the topological core of $RT_0$ which is equivalent to a *contact algebra* (**?**; **?**; **?**) and the more restricted definition of a *connection structure* (**?**).

First, we show that contrary to the models of *RCC* that are exclusively atomless (**?**), the theory $RT_0$ allows finite and infinite models. The proof of lemma 1 constructs finite models, and then infinite models must automatically exist. [TODO give reason]

**Lemma 1.** *There exist finite, non-trivial models of $RT^-$, $RT^-_{EC}$, and $RT_0$.*

*Proof.* The model $\mathscr{M}$ defined by $\langle a^*, b\rangle, \langle a^*, c\rangle \in C^{\mathscr{M}}$ (with all reflexive and symmetric tuples also contained in $C^{\mathscr{M}}$) satisfies all axioms of $RT^-$ and is of finite domain $\{a^*, b, c\}$ and hence is a finite model of $RT^-$. The model defined by $\langle a^*, b\rangle, \langle a^*, ib\rangle, \langle a^*, c\rangle, \langle a^*, ic\rangle, \langle b, ib\rangle, \langle c, ic\rangle, \langle b, c\rangle \in C^{\mathscr{M}}$ (again with all reflexive and symmetric tuples also contained in $C^{\mathscr{M}}$) with $\langle b, c\rangle, \langle c, b\rangle \in EC^{\mathscr{M}}$ satisfies all axioms of $RT^-_{EC}$ and has a finite domain $\{a^*, b, c, ib, ic\}$ and thus is a finite model of $RT^-_{EC}$. In (**?**) we proved that the Cartesian product of a finite model of $RT^-$ and a finite model of $RT^-_{EC}$ is always a finite model of $RT_0$ if each is extended by a new individual serving as suprema. Hence, the product of the presented models is a finite model of $RT_0$. □

### 3.1 Topological spaces

Trying to characterize the models of $RT_0$ using topological spaces and the common tool of separation axioms is natural since the intended models of the theory are defined over topological spaces. Here we only present the major results, see (**?**) for details. The use of separation axioms fails but shows parallels to the topological characterizations of the *RCC* and Boolean Connection Algebras in general. (**?**) characterized the models as weakly regular (a stronger form of semi-regularity) but also showed that $T_0$, $T_1$ are not forced by the axioms. For a model of $RT_0$ there always exists an embedding topological space $(X,T)$ over the set $X = \Sigma_U =_{def} \bigcup \{\Omega_{[c_n]} | c_n \in \Sigma_C\}$ and the topology $T = \Sigma_U^T = \{\emptyset\} \cup \{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\} \cup \{\bigcup Z | Z \subseteq \{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\}\}$ that satisfies $T_0$, but $T_0$ cannot generally be assumed for topological spaces constructed from models of $RT_0$. For the finite (atomic) models the embedding space is always reducible to discrete topologies and hence uninteresting. The infinite models of $RT_0$ are embeddable in semi-regular spaces (that are $T_1$ but not necessarily Hausdorff or regular) which follows from the *smooth boundaries* condition of $RT_T$ forcing all open sets to be regular open. An equivalent topological property to capture the *full interiors* condition was not found (local connectedness fails).

**Theorem 1.** *A model of $RT_0$ with infinite number of individuals can be embedded in a semi-regular topological space.*

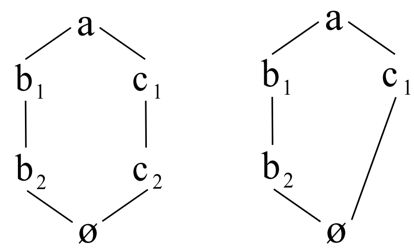
*Proof.* See (**?**). □

Notice that this theorem covers both the atomless models and the models with infinite number of atoms.

### 3.2 Lattices

The similarity between posets that underlie lattices on the one side and parthood structures as found in mereology on the other side hints a characterization of the models of $RT_0$ as lattices using the sum and intersection operations $\cup^*$ and $\cap^*$ as join and meet. The empty set which is not part of the set $Y$ of any mereotopological structure $RT_T$ has to be added as zero element to form a bounded lattice.

**Proposition 1.** *A model $\mathscr{M}$ of $RT_T$ can be represented as lattice (algebraic structure) $\mathscr{L}^{\mathscr{M}} = (Y \cup \emptyset, \cup^*, \cap^*)$ over the partial order $P^{\mathscr{M}}$: if $\langle x, y\rangle \in P^{\mathscr{M}}$ then $x \leq_{\mathscr{L}^{\mathscr{M}}} y$.*




(a) lattice $\mathscr{L}_6$ (b) pentagon sublattice $N_5$

Figure 1: Six element sublattice contained in every lattice $\mathscr{L}^{\mathscr{M}}$ and one possible pentagon sublattice

The lattice is uniquely defined for any model of $RT^-$, $RT^-_{EC}$, and $RT_0$ because it is only defined by a model's parthood extension. But a particular lattice does not necessarily represent a unique model since the extension of $EC^{\mathscr{M}}$ is not represented in the lattice. Here we only present the main results of the lattice characterization, see (**?**) for details. We use standard lattice concepts (e.g. unicomplementation and pseudocomplementation) from (**?**), supplemented by semimodularity (**?**), and orthocomplementation and orthomodularity (**?**) properties for the characterization. Using lattices we give a representation theorem for the models of $RT^-$.

One important observation is the following lemma 2 (caused by A11) which results in a special 6-element sublattice $\mathscr{L}_6$ for every model of $RT_T$ (lemma 3).

**Lemma 2.** *In any model of $RT^-_{EC}$ or $RT_0$ two non-open, non-intersecting but connected individuals must exist.*

*Proof.* See appendix. □

**Lemma 3.** *Every model $\mathscr{M}$ of $RT_T$ entails the existence of a 6-element sublattice $\mathscr{L}_6$ of $\mathscr{L}^{\mathscr{M}} = (Y \cup \emptyset, \cap^*, \cup^*, \emptyset, a^*)$ with following properties:*

*(1) $\mathscr{L}_6$ has set $Y' = \{\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{c_1}, \mathbf{c_2}, \emptyset\} \subseteq Y^{\mathscr{M}}$;*
*(2) for $n, m \in \{1,2\}$, $\mathbf{a} = \mathbf{b_n} \cup^* \mathbf{c_m}$ is the supremum of $\mathscr{L}_6$;*
*(3) for $n, m \in \{1,2\}$, $\emptyset = \mathbf{b_n} \cap^* \mathbf{c_m}$ is the infimum of $\mathscr{L}_6$;*
*(4) $\mathbf{b_1} \cap^* \mathbf{b_2} = \mathbf{b_2}$ and $\mathbf{c}_1 \cap^* \mathbf{c_2} = \mathbf{c_2}$;*
*(5) $\mathbf{b_1} \cup^* \mathbf{b_2} = \mathbf{b_1}$ and $\mathbf{c_1} \cup^* \mathbf{c_2} = \mathbf{c_1}$;*
*(6) $\mathbf{a} \cup^* x = \mathbf{a}$ and $\mathbf{a} \cap^* x = x$ for all $x \in Y'$;*
*(7) $\emptyset \cup^* x = x$ and $\emptyset \cap^* x = \emptyset$ for all $x \in Y'$.*

*Proof.* See appendix. □

By removing an arbitrary element from $\{\mathbf{b_1}, \mathbf{c_1}, \mathbf{b_2}, \mathbf{c_2}\}$ of $\mathscr{L}_6$ we obtain a sublattice $\mathscr{L}_5$ that is still closed under join and meet and is a pentagon $N_5$, compare figure 3.2. With distributivity requiring modularity which is equivalent to the absence of pentagons as sublattices, we derive following corollary.

**Corollary 1.** *No lattice associated with a model of $RT_0$ or $RT^-_{EC}$ is distributive.*

Notice that this corollary does not apply to models of $RT^-$. This result strictly separates the models of $RT_0$ from those of the *RCC* and Clarke's system. (**?**; **?**) found models of the *RCC* representable as inexhaustible (atomless) pseudocomplemented distributive lattices and models of the *Calculus of Individuals* were in (**?**) shown to be isomorphic to complete atomless Boolean algebras that are distributive lattices.

However, for the models of $RT^-$ we prove join- and meet-pseudocomplementedness as well as orthocomplementedness (using the topological complement as orthocomplement) and that the intersection of these classes of lattices is an isomorphic characterization of the models of $RT^-$.

**Theorem 2. (Representation Theorem for $RT^-$).** *The lattices arising from models of $RT^-$ are isomorphic to doubly pseudocomplemented ortholattices (p-ortholattices).*

*Proof.* See appendix. □

The models of $RT$ and $RT^-_{EC}$ are then proper subsets of the p-ortholattices that are not atomistic, not semimodular, not orthomodular, nor uniquely complemented by the existence of a sublattice $\mathscr{L}_6$ and thus $N_5$ in the resulting lattices. External connection relations are not expressed in the lattice, hence lattices alone fail to characterize models of $RT^-_{EC}$ and of the full theory $RT_0$. Nevertheless, the above representation is already helpful, since only the trivial models are not yet excluded. All properties of join- and meet-pseudocomplemented and orthocomplemented lattices can be directly applied to the models of the mereotopology.

### 3.3 Graphs

To avoid neglecting the extension $EC^{\mathscr{M}}$, we can represent a model $\mathscr{M}$ of $RT_0$ as graph $G(\mathscr{M})$ where the individuals of the model are vertices and the dyadic primitive relation $C$ is the adjacency relation of the graph.

**Proposition 2.** *A model $\mathscr{M}$ of (a subset of) $RT_0$ has a graph representation $G(\mathscr{M}) = (V,E)$ where $V_G = Y^{\mathscr{M}}$ and $\mathbf{xy} \in E_G \iff \langle \mathbf{x}, \mathbf{y} \rangle \in C^{\mathscr{M}} \iff [\![x]\!]_g \cap [\![y]\!]_g \neq \emptyset$.*

If we take as subset the theory $RT_C$, the models can be captured by the absence of true twins in their graphs. This characterization as graphs without true twins generalizes to *connection structures*. Notice although theorem 3 is not restricted to finite (or atomic) models of $RT_C$, only for the finite models of $RT_C$ is the resulting graph finite and simple.

**Definition 1.** Two vertices $x, y \in V(G)$ are true (false) twins in a graph $G$ iff $N[x] = N[y]$ ($N(x) = N(y)$).

**Theorem 3. (Representation Theorem for $RT_C$).** *The graph representations $G(\mathscr{M})$ of models $\mathscr{M}$ of $RT_C$ are isomorphic to the graphs that have no true twins.*

*Proof.* See appendix. □

A more restricted class of graphs can be defined by a vertex ordering called maximum neighborhood inclusion ordering (*mnio*) that is a special case of a maximum neighborhood ordering defining dually chordal graphs (**?**). Moreover, the graphs with *mnio* are always free of true and false twins. For definition of an *mnio* and proofs see (**?**) where every graph associated to a model of $RT$ yields an *mnio* and therefore is dually chordal and twin-free. Although *mnio*s capture important properties of parthood hierarchies, they are still not specific enough so that all graphs that yield an *mnio* are models of $RT^-$ or even $RT$.

### 3.4 Lattices as Graphs

The pure lattice-theoretic representation does not account for external connection. To overcome this, we combine the advantages of the lattice and graph representations to define graphs over lattice structures. Remember that the lattices nicely capture parthood structures and complementation whereas the graphs are able to represent full models of $RT_0$ and $RT^-_{EC}$. For a p-ortholattice we already know there exists a model of $RT^-$, now the representation of such lattice as graph allows us to model external connection.

**Proposition 3.** *Every p-ortholattice $\mathscr{L}$ over a set of elements $Y$, has a representation as undirected graph $G^{\mathscr{L}} = (V,E)$ with $V \cong Y$ and $x,y,z \in Y\,[z \leq x \wedge z \leq y] \iff xy \in E(G^{\mathscr{L}})$. $G^{\mathscr{L}}$ is finite and simple if $\mathscr{L}$ is finite.*

We observed a correlation of orthocomplements in the lattices with connectedness in the models that leads to a representation theorem for the finite models of $RT^-_{EC}$. We must restrict the theorem to the finite models since our proofs rely on the lattices being atomic. [TODO explain why only finite models].

**Theorem 4. (Representation Theorem for finite models of $RT^-_{EC}$).** *Each finite not unicomplemented p-ortholattice $\mathscr{L}$ represented as graph $G^{\mathscr{L}}$ extended by the non-empty set $E_{EC} = \{xy | y \not\leq x^{\perp}\} \setminus E(G^{\mathscr{L}})$ to a graph $(V_{G^{\mathscr{L}}}, E_{G^{\mathscr{L}}} \cup E_{EC})$ is isomorphic to a finite model of $RT^-_{EC}$.*

*Proof.* See appendix. □

As by-product, we learn that the finite models are a proper subset of Clarke's contact algebras characterized in (**?**) as complete ortholattices.

## 4 Discussion

We used three kinds of mathematical structure to characterize models of (subsets of) $RT_0$. The results using topological spaces were sparse, it especially fails to characterize the finite models beyond discrete topologies. If we represent finite models by infinite point sets, the resulting spaces are not even $T_0$ and hence from a topological stance uninteresting. If we model the finite models by finite point sets, we reduce them to trivial discrete topology.

The lattice-theoretic approach was more fruitful; characteristic properties of the models of $RT^-$ can be captured solely by orthocomplementation and pseudocomplementation which together give an isomorphic description of the models of $RT^-$ as p-ortholattices. However, there was no room for the distinctive mereotopological concepts of external connection and weak contact; lattices alone cannot account for A12 and A13. The existence of external connections prohibits unicomplemented and any kind of modular lattices from representing models of $RT^-_{EC}$. The lattices representing model of $RT_0$ are strictly not unicomplemented. Hereby, the models are delimited from those of the *Calculus of Individuals* and of the *RCC*. The former were characterized as Boolean lattices which are equivalent to the uniquely complemented distributive pseudocomplemented lattices (distributive pseudocomplemented is not enough, this class contains Heyting and Stone lattices as well) and from inexhaustible (corresponds to atomless) distributive pseudocomplemented lattices models of the *RCC* can be constructed.[2] Both theories have models with distributive, unicomplemented lattices. This is partly caused in Clarke's system by the error in the definition of external connection that maps it to overlap and in *RCC* by the lack of any distinction between open and closed elements as separate individuals of the models. This simplification in the *RCC* sacrifices a higher expressiveness offered by the system $RT_0$. Empirical approaches will be necessary to evaluate in which cases such simplification is acceptable and which applications or domains require the higher expressiveness of Asher and Vieu's theory.

A third approach represents the models uniquely as undirected graphs based on the single dyadic primitive $C$. We characterized $RT_C$ and the more generic *connection structures* as twin-free graphs. However, for this kind of twin-freeness no characteristic properties are known in graph-theory. In (**?**) we further defined a new vertex ordering called *maximum neighborhood inclusion order* (mnio) and demonstrated that this ordering defines a class of graphs that includes all graphs of $RT^-_{EC}$, and itself is a proper subset of the dually chordal graphs. These orderings are somewhat characteristic for the graphs of $RT^-_{EC}$ but not all properties defined by the axioms of $RT_0$ are captured, especially the existence of sums, intersections, and interiors is not properly translated to graphs withs *mnio*s. Therefore mnios also fail to characterize the models of $RT_0$ up to isomorphism. Nevertheless, the graph-theoretic characterization gives us valuable insight into the models of the mereotopology and their substructures and we collected in (**?**) some graph-theoretic properties that might generalize to other mereotopological theories.

Bounded lattices naturally capture the existence of sums and intersections of pairs of elements as well as the essential parthood order of mereological theories, while graphs are capable of fully representing models of $RT_0$. This led to a full characterization of the finite models of $RT^-_{EC}$ in terms of graphs of lattices: every finite not unicomplemented p-ortholattice $\mathscr{L}$ uniquely defines a graph $G^{\mathscr{L}}$ that is equivalent to a finite model $\mathscr{M}$ of $RT^-_{EC}$ where $\langle x,y\rangle \in O^{\mathscr{M}} \iff \exists z[z \leq x \wedge z \leq y \wedge z \neq \emptyset] \iff xy \in E(G^{\mathscr{L}})$ and $\langle x,y\rangle \in EC^{\mathscr{M}} \iff \{xy \in \big(E(G^{\mathscr{L}}_{EC}) \setminus E(G^{\mathscr{L}})\big) \wedge y \nleq x^{\perp}\}$. These constructs maintain ortho- and pseudocomplementation while uniquely extending the graphs to twin-free graphs with non-empty extensions $EC^{\mathscr{M}}$.

In a final step conducted in (**?**) finite models of $RT_0$ with weak contacts were constructed as direct products of finite p-ortholattices, see the proof of lemma 1 for an example. The product of two finite p-ortholattices of which at least is not unicomplemented, each extended them by separate closures of their suprema, is a (finite) model of $RT$. The direction that any model of $RT$ can be obtained in a similar fashion is still open, leading to a representation theorem of the models of full $RT_0$. Although such a theorem is desired, we think that it can give little extra insight into the models of

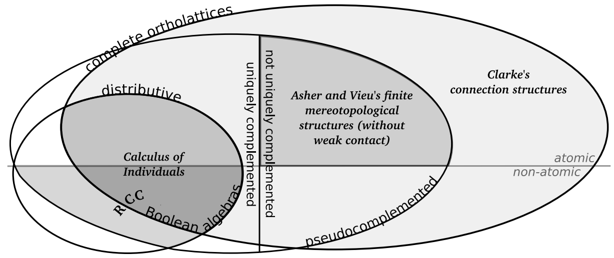



Figure 2: Asher and Vieu's mereotopology, Clarke's *Calculus of Individuals*, and the *RCC* as subclasses of lattices. There also exist models of *RCC* that are the atomless distributive pseudocomplemented lattices (but not representation theorem exists yet), the models of the *Calculus of Individuals* are the distributive ortholattices, and the models of $RT^-_{EC}$ are the not-unicomplemented, pseudocomplemented ortholattices.

$RT_0$. The representation theorem for $RT^-$ is more important and characteristic for the mereotopology. Through the given characterization it is now easy to construct p-ortholattices that correspond to models and even more importantly, we can identify the extensions of all mereotopological relations from the lattice alone. Orthocomplements in the lattices map to complements in the models, the join and meet of pairs in the lattice represent the unique sum and intersection in the corresponding model. The closure and interiors are equivalent to the meet- and join-pseudocomplements of the orthocomplement. Overlap relations produce a meet distinct from the empty set and external connections for an element are identified by all elements not part of its orthocomplement that the element itself is not connected to by any other means.

An open question for Asher and Vieu's mereotopology is whether the infinite models always give complete lattices. [3] If not, the theory $RT_0$ actually weakens Clarke's unrestricted fusion axiom. Otherwise, we obtain a proof that the unrestricted fusion can be replaced lossless by the sum axiom A5 without impacting the infinite models.

Overall, the paper outlines a methodology for characterizing models of mereotopologies to enable a model-theoretic comparison of mereotopologies in order to understand differences and commonalities between different axiomatizations. The lattice-based approach turned out most promising since it captures essential mereological and topological concepts such as parthood and complements. All mereotopological theories using a single primitive can be also represented as graphs of lattices as demonstrated. For the future, we want to analyze the system of (**?**) that explicitly distinguishes a topological (*simple region*) and a mereological primitive (*parthood*) and comprises a notion of convexity. Other ontologies not yet fully treated in a model-theoretic are the *RCC* and the mereotopology of (**?**). On the reverse one can choose a promising class of lattices and show whether it yields useful mereotopological systems -

[2]A full representation theorem for the models of *RCC* is still outstanding, but we expect all models of the *RCC* to be distributive.

[3]For the *RCC* there is no reason why non-complete lattices cannot represent models.

either generic or limited to a certain application domain. The set of potential candidates identified in (**?**) include semimodular lattices, geometric lattices, the full class of p-ortholattices, Stone lattices, Heyting lattices (compare (**?**)), the full class of pseudocomplemented distributive lattices, and - more generic - pseudocomplemented or orthocomplemented lattices.

# Appendix

For some of the proofs we need proposition 4 which is a consequence of the definition of $PP$ as irreflexive partial order.

**Proposition 4.** *For* $\mathbf{x},\mathbf{y} \in Y^{\mathscr{M}}$ *in a model* $\mathscr{M}$ *of* $RT_0$*,* $\langle \mathbf{x},\mathbf{y}\rangle \in PP^{\mathscr{M}}$ *iff* $N[\mathbf{x}] \subset N[\mathbf{y}]$ *holds in the representing graph* $G(\mathscr{M})$*.*

## Proof for lemma 2

*Proof.* Condition (vii) of $RT_T$ requires two elements $x,y \in Y$ to share a point, but no interior point ((note that $int(x \cap y) = int(x) \cap int(y)$)): $x \cap y \neq \emptyset \wedge int(x \cap y) = \emptyset$. Thus $x$ and $y$ share only boundary points. If w.l.g. $x$ is open, i.e. $x = int(x)$, it cannot contain any boundary points to share in an external connection. Thus for some $x,y$ to be externally connected, $x$ and $y$ must be non-open (but not necessarily closed). Then $x$ and $y$ cannot intersection in a common part, since this common part would have a non-empty interior (by condition (ii) of $RT_T$) and thus violate A11 or D4 in the equivalent model of $RT_0$ or $RT^-_{EC}$. □

## Proof for lemma 3

*Proof.* Since the axioms force the existence of a pair of externally connected individuals which are non-open. Let us call these $\mathbf{b_1}$ and $\mathbf{c_1}$. Because of their non-openness, two open regions $\mathbf{b_2} = int(\mathbf{b_1})$ and $\mathbf{c_2} = int(\mathbf{c_1})$ must exist as interiors according to (ii) of $RT_T$. These regions $\mathbf{b_2}$ and $\mathbf{c}_2$ are part of and connected to the element they are interior of, $\mathbf{b_1}$ and $\mathbf{c_1}$, respectively. $\mathbf{b_2}$ and $\mathbf{c_2}$ are not connected to each other in order to satisfy the condition of external connection for $\mathbf{b_1}$ and $\mathbf{c_1}$ (see D4 or (vii) of $RT_T$). This set of regions $Y'$ with $\mathbf{a} = \mathbf{b_1} \cup^* \mathbf{c_1}$ (for $\mathbf{a} = a^*$ it is actually the smallest model allowed by $RT^-_{EC}$) together with the empty set forms a sublattice with $\mathbf{a}$ as supremum, two branches consisting of $\mathbf{b_1}$ and $\mathbf{b_2} = int(\mathbf{b_1})$ respectively $\mathbf{c_1}$ and $\mathbf{c_2} = int(\mathbf{c_1})$, and the zero element $\emptyset$. Any model of $RT_T$ contains at least these elements. If the lattice contains additional elements, $\mathscr{L}_6$ always forms a sublattice of it, since the elements $\mathbf{a},\mathbf{b_1},\mathbf{b_2},\mathbf{c_1},\mathbf{c_2},\emptyset$ are closed under $\cup^*$ and $\cap^*$. Hence the axioms force any model of $RT_0$ or $RT^-_{EC}$ to have $\mathscr{L}_6$ as sublattice. □

## Proof outline for theorem 2

**Definition 2.** (**?**; **?**) Let $L$ be a lattice with infimum 0 and supremum 1.

An element $a'$ is a meet-pseudocomplement of $a \in L$ iff $a \wedge a' = 0$ and $\forall x(a \wedge x = 0 \Rightarrow x \leq a')$; $a'$ is a join-pseudocomplement of $a \in L$ if and only if $a \vee a' = 1$ and $\forall x(a \vee x = 1 \Rightarrow x \geq a')$.

*Claim* 1. Any lattice $\mathscr{L}$ constructed from a model of $RT_0$ is meet- and join-pseudocomplemented.

*Proof.* We know every such lattice $\mathscr{L}$ is complemented: for every $a \in Y \cup \emptyset$ there exists a complement $a'$ so that $a \wedge a' = 0$ and $a \vee a' = 1$. In (**?**), we proved that for a complement $a'$ of $a$, $int(a')$ and $cl(a')$ are also complements of $a$. Now we claim: (i) every element $b$ with $b > cl(a')$ has a non-zero meet with $a$ and thus cannot be meet-pseudocomplement of $a$ and every element $c$ with $c < int(a')$ has a join with $a$ that is not the supremum; (ii) every element $b$ with $a \wedge b = 0$ or $a \vee b = 1$ satisfies the condition $b \leq cl(a')$ or $b \geq int(a')$, respectively.

(i) Assume $b$ with $b > cl(a')$ and $b \wedge a = 0$ exists. Then the extension of $C$ in which $b$ participates must subsume the extension of $C$ in which $cl(a')$ participates. If the extensions of $O$ where $b$ or $cl(a')$ participate are the same then either $cl(a')$ is not closed ($b$ has an additional another external connection) or $b$ and $cl(a')$ have the same extensions of $C$ and are by A3 identical. If the extension of $O$ in which $b$ participates is strictly greater than that of $cl(a')$, then $b$ must overlap with some part of $a$ and $b \wedge a = 0$ does not longer hold. In both cases we derive a contradiction.
(ii) From (i) we know there exists no such $b$ with $b > cl(a')$ so that $b \wedge a = 0$. Now we prove that no other element $b$ exists with $b \wedge a = 0$ that is incomparable to $cl(a')$. Notice that every element $b$ is either comparable to $a$ or $-a$, see proposition 4. Assume $a'$ to be orthocomplement of $a$ (we later prove that such orthocomplement always exists). If $b$ is comparable to $a$ then obviously $a \wedge b = 0$ does not hold. Hence $b$ must be comparable to $-a$. The trivial case is $cl(a') = -a$. Otherwise the sum $b \cup^* cl(a')$ overlaps in some part(s) with $a$ ($cl(a')$ is already maximally connected to $a$ without overlap, see the argument for (i)), which in turn requires one part (either of $b$ or $cl(a')$, or of a third element) to overlap with $a$. That would mean either $b$ or $cl(a')$ overlaps with $a$ and $a \wedge b = 0$ or $a \wedge cl(a') = 0$ does not hold. Hence no such $b$ can exist. From (i) and (ii) together with the fact that $cl(a')$ is also a complement of $a$, $cl(a')$ must be the meet-pseudocomplement of $a$.
The proof for the join-pseudocomplements is analogous. □

**Definition 3.** (**?**) A bounded lattice is an ortholattice (orthocomplemented lattice) iff there exists a unary operation $^\perp : L \rightarrow L$ so that:
(1) $\forall x \left[x = x^{\perp\perp}\right]$ (involution law)
(2) $\forall x \left[x \wedge x^\perp = \perp\right]$ (complement law; or $\forall x \left[x \vee x^\perp = \top\right]$)
(3) $\forall x, y \left[x \leq y \equiv x^\perp \geq y^\perp\right]$ (order-reversing law).

*Claim* 2. Any lattice $\mathscr{L}$ constructed from a model of $RT_0$ is an ortholattice with the topological complement $\sim$ as orthocomplementation operation.

*Proof.* We check conditions (1) to (3) for the operation $\sim$, choosing $\sim a^* = \emptyset$ and $\sim \emptyset = a^*$ to make $\sim$ a complete function on the set $Y \cup \emptyset$. Property (1) and (2) ( $x \cap^* \sim x = \emptyset$) hold from the set-theoretic definition of topological complements. To prove (3), consider $x$ and $y$ as sets of points: $x \leq y$ (in the lattice) iff $x \subseteq y$. If $x = y$ then $\sim x = \sim y$ and (3) holds trivially. Hence assume $x \subset y$, then all the points in $y \setminus x$ (non-empty) must be part of the complement of $x$, i.e. $y \setminus x \subseteq \sim x$. Since all points that are both in $x$ and $y$ are in neither complement and all points in neither set are in both complements, $\sim y$ must be a proper subset of $\sim x$, i.e. $\sim x = a^* \setminus (x \cap y)$ and $\sim y = a^* \setminus (x \cap y) \setminus (y \setminus x)$. $a^* \setminus (x \cap y) \setminus (y \setminus x) \subseteq a^* \setminus (x \cap y)$ follows and with $y \setminus x$ distinct from $x \cap y$ and assumed to be non-empty: $a^* \setminus (x \cap y) \setminus (y \setminus x) \subset a^* \setminus (x \cap y)$. Thus $\sim y < \sim x$, satisfying the order-reversing law (3). □

## Proof outline for theorem 3

*Proof.* If a graph $G(\mathscr{M})$ has two vertices $x, y \in V(G(\mathscr{M}))$ with $N[x] = N[y]$, then A3 is violated unless $x = y$. On the reverse, a graph without true twins directly satisfies A3. □

## Proof outline for theorem 4

*Proof.* Notice that for every p-ortholattice $\mathscr{L}$ the graph $G^{\mathscr{L}}$ is uniquely defined because of the unique definitions of $V(G^{\mathscr{L}})$ and $E(G^{\mathscr{L}})$. Thus the graph $G^{\mathscr{L}}$ is uniquely defined for every model of $RT^-_{EC}$. Moreover, the lattices representing models of $RT^-_{EC}$ are not unicomplemented p-ortholattices, where a finite model $\mathscr{M}$ of $RT^-_{EC}$ gives a finite not-unicomplemented p-ortholattice $\mathscr{L}$ which again gives a finite graph $G^{\mathscr{L}}$ with non-empty extension $E_{EC}$. Thus every finite model of $RT^-_{EC}$ results in a graph $G^{\mathscr{L}}$ as required by the theorem.

The reverse: any graph $G^{\mathscr{L}}_{EC} = \left(V(G^{\mathscr{L}}), E(G^{\mathscr{L}}) \cup E_{EC}\right)$ constructed from a not unicomplemented p-ortholattice gives a model $\mathscr{M}$ of $RT^-_{EC}$. The extension $E_{EC} = \left\{xy | y \not\leq x^\perp\right\} \setminus E(G^{\mathscr{L}})$ is non-empty (claim 1) and thus satisfies A11. Afterwards we show that $G^{\mathscr{L}}_{EC}$ satisfies the axioms A1 to A10 and A13 (A1, A2, A4, A7 and A9 are straightforwards and omitted here).

*Claim* 1. $E_{EC} = \left\{xy | y \not\leq x^\perp\right\} \setminus E(G^{\mathscr{L}})$ is non-empty.

Assume the contrary, i.e. that $E_{EC} = \{\}$ for a graph $G^{\mathscr{L}}$. Then it holds that $\left\{xy | y \not\leq x^\perp\right\} \subseteq E(G^{\mathscr{L}})$. Additionally, $E(G^{\mathscr{L}}) \subseteq \left\{xy | y \not\leq x^\perp\right\}$ because no individual can be connected to its complement or parts thereof. But then the graph representation of each element $x$ has a unique neighborhood $N[x] = \left\{xy | y \not\leq x^\perp\right\}$ just from the parthood relation. Hence the underlying lattice is unicomplemented.

*Claim* 2. $G^{\mathscr{L}}_{EC} = G^{\mathscr{L}} \cup \left\{xy | y \not\leq x^\perp\right\}$ satisfies A3.

Assume there exist two elements $x, y \in V(G^{\mathscr{L}}_{EC})$ such that $N[x] = N[y]$. Since $xx^\perp \notin E$ and thus $x^\perp \notin N[x]$ it follows that $x^\perp \notin N[y]$. The same for $y^\perp$, i.e. $y^\perp \notin N[x]$. Then by the definition of $G^{\mathscr{L}}_{EC}$, a contradiction arises because both $y^\perp \leq x^\perp$ and $x^\perp \leq y^\perp$ must hold. Hence no two vertices $x, y \in V(G^{\mathscr{L}}_{EC})$ with $N[x] = N[y]$ can exist.

*Claim* 3. The extension $P^{\mathscr{M}}$ of the parthood relation in $\mathscr{M}$ is given by the lattice $\mathscr{L}$, i.e. $x \leq y \iff \langle x, y\rangle \in P^{\mathscr{M}}$.

Assume $x \leq y$ for some pair $x, y$. That means $N[x] \subseteq N[y]$. Whenever a third element $z$ is connected to $x$, it will also be connected to $y$, since by the order-reversing law, $y^\perp \leq x^\perp$ holds and if $z \not\leq x^\perp$ then $z \not\leq y^\perp$. So $N[x] \subseteq N[y]$ is preserved in $G^{\mathscr{L}}_{EC}$ (when adding $E_{EC}$) and thus $\langle x, y\rangle \in P^{\mathscr{M}}$. On the reverse, if $\langle x, y\rangle \in P^{\mathscr{M}}$ in a model of $RT^-_{EC}$, then $N[x] \subseteq N[y]$ in the graph $G^{\mathscr{L}}_{EC}$. If now $N[x] \not\subseteq N[y]$ in $G^{\mathscr{L}}$, then $x \not\leq y$ and $y^\perp \not\leq x^\perp$. Some $z$ exists with $\langle x, z\rangle \in E_{EC}$ but $\langle y, z\rangle \notin E_{EC}$. Then either $y^\perp > x^\perp$ or $y^\perp$ and $x^\perp$ are incomparable with the consequence of $z \in N[x]$ but $z \notin N[y]$ or $y^\perp \in N[x]$ but $y^\perp \notin N[y]$, respectively.

*Claim* 4. $G^{\mathscr{L}}_{EC} = G^{\mathscr{L}} \cup \left\{xy | y \not\leq x^\perp\right\}$ satisfies A5.

Let $z \in V(G^{\mathscr{L}})$ be the sum element of some pair of elements $x, y \in V(G^{\mathscr{L}})$ with $z = x \cup^* y \geq x, y$. We prove each direction of the equivalence in A5 individually.

(a) $\exists v \left[(C(v, x) \vee C(v, y)) \rightarrow C(v, z)\right]$
Since $z \geq x$, either $z = x$ and $zv \in E \iff xv \in E$ or $z > x$ and by proposition 4 $xv \in E \Rightarrow zv \in E$; the same for $y$.

(b) $\exists v \left[(C(v, x) \vee C(v, y)) \leftarrow C(v, z)\right]$
Assume there exists an element $v$ s.t. $zv \in E$ but $xv, yv \notin E$.

Let $v$ be comparable to $z$ but not to $x$ and $y$. This can only occur if $v < z$ and $v$ is disjoint with both $x$ and $y$. If there is a common proper part $u$, i.e. w.l.g. $u < v, x$ then $v$ and $x$ are connected. If no such $u$ exists, there exists at least three atoms in this subbranch of the lattice. But then the lattice is not pseudocomplemented, since dual-atoms not comparable to these atoms would not have unique join-pseudocomplements. Otherwise if $v$ is comparable to $z$, it is comparable to at least one of $x$ and $y$.
If $v$ is not comparable to $z$, then $v$ is comparable to $z^\perp$, i.e. either $v \leq z^\perp$ or $v > z^\perp$. In the first case $v$ cannot be connected to $z$ by definition contrary to the assumption. In the latter case $v$ is comparable to one of $x^\perp$ and $y^\perp$. If $v$ would be incomparable to both, there must exist three distinct dual-atoms in this subbranch of the lattice and the lattice is not meet-pseudocomplemented. If $v$ is comparable to only one of them, i.e. w.l.g. to $x$ then $yv \in E$ since $v \nleq y^\perp$. If $v$ is comparable to $x$ and $y$ and $v < x^\perp, y^\perp$ then $v = x \cap^* y$ and thus $v^\perp = x \cup^* y$ by the order-reversing law. Hence $z$ is not the sum of $x$ and $y$. If $v > x^\perp, y^\perp$ (note that if $x$ and $y$ are comparable with each other, they are ordered and $z$ is not the sum of $x$ and $y$) then $v > z^\perp$ and $xv, yv, zv \notin E$ would follow contrary to our assumption that $zv \in E$.
*Claim* 5. $G^{\mathscr{L}}_{EC} = G^{\mathscr{L}} \cup \{xy | y \nleq x^\perp\}$ satisfies A6.

By claim 3 the parthood and hence the overlap relation is predefined by the lattice. We show that if the intersection $z = x \cap^* y$ given by the lattice $\mathscr{L}$ with $z < x, y$ has an additional element $v \in N(z)$, then $v \in N(x), N(y)$: assume $v$ with $zv \in E$, then $v \nleq z^\perp$. Since $z^\perp \geq x^\perp, y^\perp$ it follows that $v > x^\perp, y^\perp$ or $v$ is incomparable to $x^\perp, y^\perp$. The latter case also implies $v \nleq x^\perp$ and $v \nleq x^\perp$. Thus in any case, $vx, vy \in E$.
*Claim* 6. $G^{\mathscr{L}}_{EC} = G^{\mathscr{L}} \cup \{xy | y \nleq x^\perp\}$ satisfies A8.

For any $x$ take the greatest open element $y$ with the same overlap extension and $y \leq x$. Such an element must exist, since the underlying lattices are atomic: any atom $y < x$ satisfies A8 because it is not externally connected: its orthocomplement $y^\perp$ is a dual-atom (by orthocomplementation) and $\forall z \left[z \ngeq y \rightarrow y^\perp \geq z\right]$ and for all such $z$, $yz \notin E$ follows.
*Claim* 7. $G^{\mathscr{L}}_{EC} = G^{\mathscr{L}} \cup \{xy | y \nleq x^\perp\}$ satisfies A10.

By D8 $\langle x \rangle OP^{\mathscr{M}}$ for a model $\mathscr{M}$ associated to $G^{\mathscr{L}}_{EC}$ iff $\{xv | v \nleq x^\perp\} = \{\}$ (similar for $y$). Then $\neg\exists v \left[v \leq x^\perp, y^\perp | xv \in E_{EC} \textit{ or } yv \in E_{EC}\right]$ and with $z = x \cap^* y \leq x, y$, $\{\langle z, v \rangle \in E_{EC}\} \subseteq \{\langle x, v \rangle \in E_{EC}\}, \{\langle y, v \rangle \in E_{EC}\} \subseteq \{\}$ follows for all $v \in Y$, i.e. $z$ is not externally connected. Then $\neg\exists v \left[v \leq z^\perp | zv \in E\right]$ and $z \in OP^{\mathscr{M}}$ because $zz^\perp \notin E$ and $z^\perp \geq x^\perp, y^\perp$. If $z = x$ (or $z = y$) then $y < x$ (or $x < y$) and again $z \in OP^{\mathscr{M}}$.
*Claim* 8. $G^{\mathscr{L}}_{EC} = G^{\mathscr{L}} \cup \{xy | y \nleq x^\perp\}$ satisfies A13.

A13 is violated iff there exist two incomparable open elements $y$ and $z$ with $y, z \geq x$. Notice that all of $x, y, z$ are related to the same set of elements by an overlap relation, otherwise this branch of the lattice contains two atoms and the lattice would not be pseudocomplemented. Then, since $y$ and $z$ are not externally connected $N[y] = N[z]$ follows which contradicts twin-freeness of $G^{\mathscr{L}}_{EC}$. Hence, A13 is satisfied in every graph $G^{\mathscr{L}}_{EC}$.

All claims together prove the representation theorem. □